Itô Calculus and Complex Brownian Motion

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Abstract

The course is divided into three chapters of approximately equal length. In the first chapter we develop the theory of the (real) Itô integral for continuous semimartingales. Then, in the second chapter we extend this theory into $\mathbb{C}$, where we use it to prove the conformal invariance and related path properties of complex Brownian motion. Finally, in the third chapter we investigate the winding and tangling of complex Brownian motion. The course finishes with a beautiful probabilistic proof of Picard's (Little) Theorem.

Introduction

Let us introduce the course back to front; beginning with definitions of the two objects that are central to the latter part of this course, namely analytic functions and complex Brownian motion.

Firstly, a $\mathbb{C}$ valued stochastic process $(Z_t)_{t\geqslant 0}$ is a **complex Brownian motion** if it can be written as

$$Z_t = X_t + iY_t \tag{0.1}$$

where $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ are independent real Brownian motions.

Secondly, let D be an open subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is said to be **analytic** if the limit

$$f'(z) = \lim_{w\to z} \frac{f(w) - f(z)}{w - z} \tag{0.2}$$

exists for all $z \in D$. An analytic function $f : \mathbb{C} \to \mathbb{C}$ is said to be an **entire** function.

Brownian motion and analytic functions are, independently, remarkable objects. They both give rise to rich and varied theories of we are able to cover only a small part. At first glance analytic functions may appear unrelated to Brownian motion, since the former are smooth objects and latter is decidedly rough. It turns out that there is a surprising connection between the two, namely the conformal invariance of Brownian motion; the image of a Brownian motion under an analytic function is a again a Brownian motion, but running at a random speed.

We say that a function $f : D \to \mathbb{C}$ **omits** the point $z \in \mathbb{C}$ if $z \notin f(D)$.

Theorem (Picard) *A non-constant entire function omits at most a single point.*

This course will end with a proof of Picard's Theorem. What is not obvious is that Picard's Theorem has a beautiful proof based on the conformal invariance and path properties of complex Brownian motion!

Whilst exploring complex Brownian motion we will make use of the full force of Itô integration with respect to continuous semimartingales. However, we do not assume any prior knowledge of Itô calculus (although it will naturally help to have some) and we cover this machinery, in $\mathbb{R}$, right from scratch, as the first part of the course.

The course has been designed to be as accessible as is possible. We assume only a basic knowledge of complex analysis, to the level of understanding the statements of the major theorems. From probability we require that reader is familiar with real Brownian motion and has some experience of martingales, even if only in discrete time.

Additional Material

The course comes with six problem sheets of approximately equal length. The difficulty of the questions varies widely and (deliberately) no indication is offered as to which questions are easy or hard. No solutions are provided, although some of the problems are standard results that can be found in books.

The problem sheets are accessible after completing sections from the course according to the following table.

Sheet 1	Section 1.4
Sheet 2	Section 1.7
Sheet 3	Section 2.2
Sheet 4	Section 2.2
Sheet 5	Section 2.5
Sheet 6	Section 2.5

Conditional on the prerequisites outlined above the course is self contained, although on some occasions we will choose to only sketch a technical proof. For general reference and further reading I recommend the following.

- For (real) Itô calculus, the latter chapters of both Ethier and Kurtz (1986) and Volume II of Rogers and Williams (2000) between them cover a vast amount of material. For Itô calculus of processes with jumps, see Chapter I of Volume II of Rogers and Williams (2000). Alternatively, both Chapter III of Ethier and Kurtz (1986) and McKean (1969) offer compact presentations of Itô calculus.

- For a detailed introduction to martingale theory, see Chapter II of volume I of Rogers and Williams (2000).

- For Brownian motion (in all dimensions), Mörters and Peres (2010) give a comprehensive account of the modern theory.

- For material related to Picard's Theorem and a probabilistic introduction to Nevanlinna Theory, the reader is directed towards Davis (1979) and the references therein.

- Priestley (2003) provides an introduction to complex analysis. For a comprehensive reference covering both real and complex analysis there is Rudin (1987)

Much of the course was constructed from the above sources along with my own lecture notes. I am grateful in all cases for the clear and careful style in which the material was presented.

Contents

Chapter 1

Martingales and Itô Calculus (in $\mathbb{R}$)

In this chapter we develop the theory of Itô calculus. We assume that you have, in some form, encountered martingales before. However, we are concious you may have only seen special cases (e.g. discrete time martingales), whereas for later chapters the full force of Itô integration with respect to continuous semimartingales is needed.

We will typically not concern ourselves with the underlying probability space and associated regularity issues. For the duration of this course we work over a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the 'usual conditions' of Rogers and Williams (2000)[1]. In this chapter we intend all stopping times, adapted processes and so on to be with respect to $(\mathcal{F}_t)$.

Recall that a stochastic process M is said to be square integrable if $\mathbb{E}\left[|M_t|^2\right] < \infty$ for all $t \geqslant 0$. If $\sup_{t\geqslant 0} \mathbb{E}\left[|M_t|^p\right] < \infty$ then we say M is bounded in L^p, where $p \geqslant 1$. Similarly, M is said to be bounded if there exists (deterministic) $C \in \mathbb{R}$ such that $\mathbb{P}\left[\forall t \geqslant 0, |M_t| \leqslant C\right] = 1$. We say that a stochastic process is continuous if its paths are continuous; similarly for left and right continuity.

In this chapter (and only in this chapter) we take all processes to be real valued.

1.1 Martingales

We will make regular use of martingale theory, although we require it only in the special case of continuous martingales and only in continuous time.

Definition 1.1.1 *A stochastic process $(M_t)_{t\geqslant 0}$ is a* ***martingale*** *if*

1. *$M_t \in \mathcal{F}_t$ for all $t \geqslant 0$,*
2. *$\mathbb{E}[M_{t+s} \,|\, \mathcal{F}_t] = M_t$ for all $s, t \geqslant 0$.*

If the first condition holds and the second is replaced by $\mathbb{E}[M_{t+s} \,|\, \mathcal{F}_t] \geqslant M_t$ then we say M is a ***submartingale****. Similarly if the first condition holds and the second is replaced by $\mathbb{E}[M_{t+s} \,|\, \mathcal{F}_t] \leqslant M_t$ then we say M is a* ***supermartingale****.*

Of course, our canonical example of a martingale is Brownian motion itself.

Lemma 1.1.2 *If M is a martingale then both M^2 and $|M|$ are submartingales.*

[1] There is no need to worry if you don't understand what this regularity conditions means. For details see of Rogers and Williams (2000).

PROOF: This is a consequence of the conditional Jensen inequality; both $f(x) = x^2$ and $f(x) = |x|$ are convex, so in both cases $\mathbb{E}[f(M_{t+s}) \,|\, \mathcal{F}_t] \geqslant f\left(\mathbb{E}\left[M_{t+s} \,|\, \mathcal{F}_t\right]\right) = f(M_t)$. ∎

We do not have time to explore martingale theory in its own right and instead we collect together the tools that we need in future chapters. We state them without proof, on the tentative assumption that you will have seen at least similar results elsewhere. Firstly, two important martingale inequalities.

Lemma 1.1.3 (Maximal inequality) *Let M be a right continuous submartingale. Then for all $x > 0$ and $T > 0$*

$$\mathbb{P}\left[\sup_{t\leqslant T} M_t \geqslant x\right] \leqslant \frac{1}{x}\mathbb{E}\left[M_T^+\right].$$

Lemma 1.1.4 (L^p inequality) *Let M be a right continuous submartingale. Then for all $p > 1$ and all $T > 0$,*

$$\mathbb{E}\left[\sup_{t\leqslant T} |M_t|^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_T|^p\right].$$

Remark 1.1.5 *The assumption of right continuity is mostly not an obstacle, see Theorem II.67.7 in Rogers and Williams (2000). It will not bother us since in later chapters we need only continuous martingales.*

Secondly, we need the continuous time version of the optional stopping theorem. Recall that a random variable τ taking values in $[0, \infty]$ is a stopping time (with respect to $(\mathcal{F}_t)$) if $\{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \geqslant 0$. Recall also that, if M is a martingale and τ is a stopping time, then the stopped process $t \mapsto M_t^\tau = M_{t\wedge\tau}$ is a martingale.

Theorem 1.1.6 (Optional Stopping) *Let M be a right continuous martingale and let $\tau_1 \leqslant \tau_2$ be stopping times. Then for each $t > 0$, $\mathbb{E}\left[M_{\tau_2\wedge t} \,|\, \mathcal{F}_{\tau_1}\right] = M_{\tau_1\wedge t}$.*

If, additionally, $\{M_{t\wedge\tau_2}\ t \geqslant 0\}$ is uniformly integrable, $\mathbb{E}\left[|M_{\tau_2}|\right] < \infty$ and $\tau_2 < \infty$ almost surely then $\mathbb{E}\left[M_{\tau_2} \,|\, \mathcal{F}_{\tau_1}\right] = M_{\tau_1}$.

If M is not a martingale but is only a sub- (resp. super-) martingale then we replace $=$ with $\geqslant$ (resp. $\leqslant$).

Remark 1.1.7 *The conditions for optional stopping in discreet and continuous time are genuinely different; applying the optional stopping in continuous time requires more care.*

By far the most useful case of the above theorem is when M or $M_{\cdot\wedge\tau}$ is a bounded martingale and thus the uniform integrability condition is automatic. Let us give an important application of the optional stopping theorem.

Example 1.1.8 *Let (B_t) be a Brownian motion with $B_0 = 0$. It is straightforward (see Question 1 on Problem Sheet 1) to show that $B_t^2 - t$ is a martingale. Let $R > 0$ and set $\tau_R = \inf\{t > 0\,;\, |B_t| \geqslant R\}$. Note that $\tau_R < \infty$ almost surely. Since B is continuous, $|B_{\tau_R}| = R$ so as $\mathbb{E}\left[B_{\tau_R}^2\right] = R^2$. Further, also by continuity of B we have $|B_{t\wedge\tau_R}| \leqslant R$ so we can apply the optional stopping theorem and deduce that $\mathbb{E}\left[\tau_R\right] = R^2$.*

Thirdly, we will need the martingale convergence theorem.

Theorem 1.1.9 (Martingale Convergence) *Let M be a right continuous supermartingale.*

1. *Suppose that* $\sup_{t\geqslant 0}\mathbb{E}\left[M_t^-\right] < \infty$. *Then there exists a random variable* M_∞ *such that* $M_t \to M_\infty$ *almost surely as* $t \to \infty$.

2. *Suppose that* $\{M_t\,;\,t \geqslant 0\}$ *is uniformly integrable. Then there exists a random variable* M_∞ *such that* $M_t \to M_\infty$ *almost surely and* $\mathbb{E}\left[|M_t - M_\infty|\right] \to 0$.

Once again, the most useful case is will be when M is a bounded martingale. Note that the first statement holds for any positive (right continuous) supermartingale.

Finally, we need the bracket process.

Theorem 1.1.10 *Let* M *and* N *be continuous square integrable martingales. Then there is a unique continuous adapted process* $\langle M, N\rangle$ *with locally finite variation such that* $\langle M, N\rangle_0 = 0$ *and* $M_tN_t - \langle M, N\rangle_t$ *is a martingale.*

We will use the standard notation $\langle M\rangle = \langle M, M\rangle$. The operator $\langle \cdot, \cdot\rangle$ does have an explicit formula, although it is rarely useful in practice:

$$\langle M, N\rangle_t = \lim_{n\to\infty}\sum_{i=1}^{n}(M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \tag{1.1}$$

where $D = \{0 = t_1 < \ldots < t_n = t\}$ and $m(D) = \max|t_{i+1} - t_i| \to 0$ as $n \to \infty$. The limit exists in the sense of Skorodhod convergence in probability on compact intervals. Note in particular that $\langle M\rangle$ is an increasing process.

The fact that $\langle M, N\rangle$ has locally finite variation (i.e. finite variation over finite time intervals) means that we can construct Lebesgue-Stieltjes integrals with respect to it! In particular, if $s \mapsto F_s$ is a stochastic process then the process

$$\int_0^{\cdot} F_s d\langle M\rangle_s$$

is well defined (providing that, $s \mapsto F_s$ almost surely satisfies appropriate integrability conditions e.g. if F is bounded).

It is easily seen that $\langle \cdot, \cdot\rangle$ is bilinear, so as in particular

$$\langle M, N\rangle = \frac{1}{4}\left(\langle M + N\rangle - \langle M - N\rangle\right). \tag{1.2}$$

We have already remarked that $B_t^2 - t$ is a martingale, where (B_t) is Brownian motion. It follows from Theorem 1.1.10 that Brownian motion has bracket process $\langle B\rangle_t = t$. In Question 1 of Problem Sheet 2 we calculate $\langle B\rangle$ using (1.1), giving an alternative proof that $\langle B\rangle_t = t$.

If M and N are independent martingales (with the same filtration) then it is easily seen that MN is also a martingale and, consequently, in this case $\langle M, N\rangle = 0$. The converse does not hold in general.

1.2 Itô Calculus I

In this section we establish our first version of the Itô integral. The class of processes that we will use as integrators is

$$\mathcal{M}_0^2 = \left\{M\,;\,(M_t)_{t=0}^{\infty} \text{ is a continuous martingale, } M_0 = 0 \text{ and } \sup_t \mathbb{E}\left[M_t^2\right] < \infty\right\}, \tag{1.3}$$

the space of L^2 bounded martingales. Given some $M \in \mathcal{M}_0^2$, the set of processes that we will be able to integrate against M is

$$\mathbb{L}^2(M) = \left\{ F\,;\, (F_t)_{t=0}^\infty \text{ is progressively measurable and } \mathbb{E}\left[\int_0^\infty F_s^2 d\langle M\rangle_s\right] < \infty \right\}. \tag{1.4}$$

If $s \mapsto F_s$ is both adapted and right (or left) continuous then $s \mapsto F_s$ is progressively measurable. This will hold for all processes considered in Chapters 2 and 3.

The linear spaces $\mathcal{M}_0^2$ and $\mathbb{L}^2(M)$ are both Hilbert spaces with norms given by

$$||M||_{\mathcal{M}_0^2} = \sup_t \mathbb{E}\left[M_t^2\right], \quad ||F||_{\mathbb{L}^2(M)} = \int_0^\infty \mathbb{E}\left[F_s^2\right] d\langle M\rangle_s$$

respectively.

A set $D = (t_i)_{i=0}^n$ is said to be a **partition** if $0 = t_0 < t_1 < \ldots < t_n < \infty$. An adapted process $(F_t)_{t=0}^\infty$ is said to be a **simple process** if it is a bounded process of the form

$$F_t = \sum_i a_i \mathbb{1}\{t \in [t_i, t_{i+1})\} \tag{1.5}$$

where $D = (t_i)_0^n$ is a partition and, for each i, a_i is a bounded $\mathcal{F}_{t_i}$ measurable random variable (note that this makes F previsible). Let $\mathbb{L}_0$ denote the set of simple processes and note that $\mathbb{L}_0 \subseteq \mathbb{L}^2(M)$ for any $M \in \mathcal{M}_0^2$. In fact, more is true.

Lemma 1.2.1 *Let $M \in \mathcal{M}_0^2$. Then $\mathbb{L}_0$ is a dense subset of $\mathbb{L}^2(M)$.*

PROOF: We omit the proof (which is best done via a monotone class argument). ■

For a simple process $F \in \mathbb{L}_0$, as in (1.5), we can define explicitly what their Itô integral is: the process $(I(F)_t)_{t=0}^\infty$ where

$$I(F)_t = \sum_i F_{t_i}\left(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}\right). \tag{1.6}$$

Note that the process value F_{t_i} associated to the increment $M_{t_{i+1}} - M_{t_i}$ is taken from time t_i, at the start of the interval $[t_i, t_{i+1})$. This is crucial in what follows, in particular for Lemmas 1.2.4 and 1.2.5.

Remark 1.2.2 *Replacing F_{t_i} with, for example, $F_{t_{i+1}}$ in (1.6) results in different integral. The most common example of such a modification is known as the Stratonovich integral, in which one takes $F_{\widetilde{t}_i}$ where $\widetilde{t}_i = \frac{t_{i+1}+t_i}{2}$. In this course we focus only on the Itô integral (1.6).*

Our construction of the Itô integral will rely on the following abstract theorem.

Theorem 1.2.3 *Let X be a metric space and let Y be a complete metric space. Let A be a dense subset of X and suppose that $f : A \to Y$ is uniformly continuous. Then there is a unique continuous map $\overline{f} : X \to Y$ such that $f = \overline{f}$ on A.*

We seek to apply the above theorem with $A = \mathbb{L}_0$, $X = \mathbb{L}^2(M)$ and $Y = \mathcal{M}_0^2$. The map f will be the map $F \mapsto I(F)$ and its closure $\overline{f}$ is our definition of the Itô integral. This leaves us with some work to do. Namely, we must show that $I(F) \in \mathcal{M}_0^2$ for all $F \in \mathbb{L}_0$ and that the map $F \mapsto I(F)$ is uniformly continuous on $\mathbb{L}_0$. We will now approach these two points.

Lemma 1.2.4 *Let $M \in \mathcal{M}_0^2$. For each $F \in \mathbb{L}_0$, $I(F)$ is a martingale.*

PROOF: Note that it follows trivially from (1.6) that $I(F)$ is a continuous square integrable (in fact, bounded) martingale. We note that for $s \leqslant t$,

$$\begin{aligned}\mathbb{E}\left[I(F)_t \,|\, \mathcal{F}_s\right] &= \mathbb{E}\left[\sum_j F_{t_j}(M_{t_{j+1}} - M_{t_j}) \,\middle|\, \mathcal{F}_s\right] \\ &= \sum_i \mathbb{E}\left[F_{t_j}\mathbb{E}\left[M_{t_{j+1}} - M_{t_i} \,|\, \mathcal{F}_{t_j}\right] \,|\, \mathcal{F}_s\right] \\ &= 0\end{aligned}$$

since M is a martingale. Note that here we abuse notation slightly and sum over the partition $(t_j) = \{s, t_{i+1}, \ldots, t_{i'}, t\}$ where $s \in (t_i, t_{i+1}]$ and $t \in (t_{i'}, t_{i'+1}]$. ■

Lemma 1.2.5 *Let $M \in \mathcal{M}_0^2$. For each $F \in \mathbb{L}_0$ the process $I(F)_t^2 - \int_0^t F_s^2 d\langle M\rangle_s$ is a martingale. Hence, the bracket process of $I(F)$ is $\int_0^\cdot F_s d\langle M\rangle_s$.*

PROOF: Let $s < t$. In similar style to the proof of Lemma 1.2.4, over the same indices but with two (identical) partitions (t_j) and (t_k), we note that

$$\begin{aligned}&\mathbb{E}\left[I(F)_t^2 - I(F)_s^2 - \int_s^t F_u^2 d\langle M\rangle_u \,\middle|\, \mathcal{F}_s\right] \\ &\qquad = \mathbb{E}\left[\big(I(F)_t - I(F)_s\big)^2 - \int_s^t F_u^2 d\langle M\rangle_u \,\middle|\, \mathcal{F}_s\right] \\ &\qquad = \mathbb{E}\left[\sum_k\sum_j F_{t_k}F_{t_j}(M_{t_{k+1}} - M_{t_k})(M_{t_{j+1}} - M_{t_j}) - \sum_k F_{t_k}^2(\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k}) \,\middle|\, \mathcal{F}_s\right]\end{aligned}$$

If we condition the summands in the first sum above on $\mathcal{F}_{t_k \wedge t_j}$ they vanish unless $k = j$ (as in the proof of Lemma 1.2.4 because M is a martingale). We thus obtain that the above is equal to

$$\mathbb{E}\left[F_{t_k}^2\,\mathbb{E}\left[\sum_k (M_{t_{k+1}} - M_{t_k})^2 - (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k}) \,\middle|\, \mathcal{F}_{t_k}\right] \,\middle|\, \mathcal{F}_s\right].$$

Noting that M is a square integrable martingale, from Theorem 1.1 the above is zero. ■

Lemma 1.2.5 implies something even better than uniform continuity of $F \mapsto I(F)$. The martingale property implies that $\mathbb{E}\left[I(F)_t^2\right] = \mathbb{E}\left[\int_0^t F_s^2 d\langle M\rangle_s\right]$ and, since this is an increasing function of t we have that

$$\int_0^\infty \mathbb{E}\left[F_s^2\right] d\langle M\rangle_s = \sup_t \mathbb{E}\left[I(F)_t^2\right]. \tag{1.7}$$

Therefore, $I(F) \in \mathcal{M}_0^2$ and the map $F \mapsto I(F)$ is an isometry. Coupled with Lemma 1.2.4 we are now in a position to apply Theorem 1.2.3.

Definition 1.2.6 *Let $M \in \mathcal{M}_0^2$. The closure in $(\mathbb{L}^2(M), \mathcal{M}_0^2)$ of the map $F \mapsto I(F)$ defines a map*

$$F_\cdot \mapsto \int_0^\cdot F_s dM_s$$

*known as the **Itô integral** with respect to M.*

Remark 1.2.7 *Alert readers will have spotted that* $\mathbb{L}^2$ *and* $\mathcal{M}_0^2$ *are really only Hilbert spaces after we quotient them by indistinguishability. This is no problem; Itô integrals are, like many other objects, only defined up to indistinguishability. We will usually not trouble ourselves to write 'almost surely' when it is needed only for this reason. We will also not trouble ourselves with the associated technicalities, save for remarking that if* F *or* M *is evanescent then* $\int_0^\cdot F_s dM_s$ *is also evanescent.*

Remark 1.2.8 *Definition* (1.2.6) *is the preferred 'modern' definition of the Itô integral, because it provides the best starting point for stochastic calculus. It is equally possible (and arguably more elegant) to define the same object implicitly using Hilbert space methods, see Section IV.28 in Rogers and Williams (2000).*

We will need integrals of the form $\int_s^t F_u dM_u$ and, in analogy with the finite variational theory, we define

$$\int_s^t F_u dM = \int_0^t F_u dM_u - \int_0^s F_u dM_u. \tag{1.8}$$

It is easily seen that the same process $t \mapsto \int_s^t F_u dM_u$ is obtained from using the same construction as above but replacing time 0 by time $s \geqslant 0$. We will use this fact in what follows without comment, although for simplicity we will usually state our results for the Itô integral as a process over $[0, \infty)$.

One well known property of the Itô integral that comes for free with our construction is the following.

Theorem 1.2.9 (Itô isometry) *Let* $M \in \mathcal{M}_0^2$. *For all* $F \in \mathbb{L}^2(M)$ *and all* t,

$$\mathbb{E}\left[\left(\int_0^t F_s dM_s\right)^2\right] = \int_0^t \mathbb{E}\left[F_s^2\right] d\langle M \rangle_s.$$

Proof: We saw in (1.7) that the relation $||I(F_t)||_{\mathcal{M}_0^2} = ||F||_{\mathbb{L}^2(M)}$ holds on the dense subset $\mathbb{L}_0$ of $\mathbb{L}^2(M)$. By continuity of the Itô integral it holds on $\mathbb{L}^2(M)$. ■

1.3 Properties of the Itô Integral I

In some ways the Itô integral behaves like the 'usual' Lebesgue-Stieltjes integral, but in other ways it does not. As usual, the easiest property to see is linearity, which we prove in the following lemma and will subsequently use without comment.

Lemma 1.3.1 *Let* $M \in \mathcal{M}_0^2$ *and* $F, G \in \mathbb{L}^2(M)$, $\alpha \in \mathbb{R}$. *Then*

$$\int_0^\cdot F_s + \alpha G_s \, dM_s = \int_0^\cdot F_s dM_s + \alpha \int_0^\cdot G_s dM_s.$$

Proof: Let $\alpha \in \mathbb{R}$, $F, G \in \mathbb{L}^2(M)$ and (by Lemma 1.2.1 let $F(n)$ and $G(n)$ be sequences of simple processes such that $F(n) \to F$ and $G(n) \to G$ in $\mathbb{L}^2(M)$. It is easily seen from (1.6) that

$$I(F(n)) + \alpha I(G(n)) = I(F(n) + \alpha G(n)). \tag{1.9}$$

Since $\mathbb{L}^2(M)$ is a linear space we have $\alpha G(n) \to \alpha G$, $F(n) + \alpha G(n) \to F + \alpha G$ in $\mathbb{L}^2(M)$. Since the Itô integral is continuous, $I(F(n)) \to I(F)$, $\alpha I(G(n)) \to \alpha I(G)$ and $I(F(n) + \alpha G(n)) \to I(F + \alpha G)$ in $\mathcal{M}_0^2$. Taking limits in (1.9) gives the result. ■

Remark 1.3.2 *Of course, the above proof works in greater generality; a continuous extension of a densely defined continuous linear operator is necessarily linear.*

For now we only need to deduce one further property of the Itô integral (and we will return to looking at other properties in Section 1.6). In particular, note that both Theorem 1.2.9 and Lemma 1.2.5 suggest that the bracket process of $\int_0^\cdot F_s dMs$ is $\int_0^\cdot F_s^2 d\langle M\rangle_s$, which is correct, but it will take us the rest of this section to prove it.

There is a standard technique for deducing basic properties of the Itô integral that is not dissimilar to its equivalents for Lebesgue-Stieltjes integrals and conditional expectations. Essentially, the program is the following:

1. Deduce that the desired property holds for simple processes.
2. Use Lemma 1.2.1 and a limit theorem (e.g. Theorem 1.3.3 below and/or dominated convergence) to carry the property onto the Itô integral of Definition 1.2.6.

The details are different in each case and often specific bounds or identities are require to justify the limit taking.

Theorem 1.3.3 *Let $M \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M)$ and suppose $(F(n)) \subseteq \mathbb{L}^2(M)$ is such that $F(n) \to F$. Then there exists a subsequence $(F(i_n))$ of $(F(n))$ such that for each $T > 0$,*

$$\sup_{t\in[0,T]} \left| \int_0^t F(i_n)_s dM_s - \int_0^t F_s dM_s \right| \to 0 \tag{1.10}$$

almost surely and in $L^2(\mathbb{P})$.

Proof: We have $\mathbb{E}\left[\int_0^\infty (F(n)_s - F_s)^2 d\langle M\rangle_s\right] \to 0$ as $n \to \infty$. By passing to a subsequence (which we do not notate) we may assume that

$$\sum_n \mathbb{E}\left[\int_0^\infty (F(n)_s - F_s)^2 d\langle M\rangle_s\right]^{1/2} < \infty. \tag{1.11}$$

For each $T > 0$, using that norms in $L^1(\mathbb{P})$ are controlled by norms in $L^2(\mathbb{P})$, followed by Lemma 1.1.4 and Theorem 1.2.9 we have

$$\begin{aligned}
\mathbb{E}&\left[\sum_n \sup_{t\in[0,T]} \left| \int_0^t F(n)_s dM_s - \int_0^t F_s dM_s \right|\right] \\
&\leqslant \sum_n \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t F(n)_s - F_s \, dM_s \right|^2\right]^{1/2} \\
&\leqslant \sum_n 2\mathbb{E}\left[\left| \int_0^T F(n)_s - F(n)_s \, dM_s \right|^2\right]^{1/2} \\
&= 2\sum_n \mathbb{E}\left[\int_0^T (F(n)_s - F_s)^2 d\langle M\rangle_s\right]^{1/2}.
\end{aligned}$$

By (1.11) the above is finite almost surely. In particular,

$$\sum_n \sup_{t\in[0,T]} \left| \int_0^t F(n)_s dM_s - \int_0^t F_s dM_s \right| < \infty$$

almost surely, which implies that almost sure convergence holds for (1.10). Further, for each $T > 0$, using Lemma 1.1.4 and Theorem 1.2.9 we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t F(n)_s dM_s - \int_0^t F_s dM_s\right|^2\right]
&\leqslant 4\mathbb{E}\left[\left|\int_0^T F(n)_s - F_s\, dM_s\right|^2\right] \\
&= 4\mathbb{E}\left[\int_0^T (F(n)_s - F_s)^2 d\langle M\rangle_s\right]
\end{aligned}
$$

which tends to zero as $n \to \infty$. Hence $L^2(\mathbb{P})$ convergence holds for (1.10). ■

Lemma 1.3.4 (Kunita-Watanabe Inequality) *Let $M, N \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M), G \in \mathbb{L}^2(N)$. Then*

$$\int_s^t |F_u G_u|\ |d\langle M, N\rangle_u| \leqslant \left(\int_s^t F_u^2 d\langle M\rangle_u\right)^{1/2} \left(\int_0^t G_u^2 d\langle N\rangle_u\right)^{1/2}$$

and

$$\mathbb{E}\left[\int_s^t |F_u G_u|\ |d\langle M, N\rangle_u|\right] \leqslant \mathbb{E}\left[\left(\int_s^t F_u^2 d\langle M\rangle_u\right)\right]^{1/2} \mathbb{E}\left[\left(\int_0^t G_u^2 d\langle N\rangle_u\right)\right]^{1/2}.$$

Remark 1.3.5 *This is a statement about finite variational integrals.*

Proof: Let us prove the first inequality. First note that the measures $|d\langle M, N\rangle|$ and $d\langle M, N\rangle$ are mutually absolutely continuous (as measures on $\Omega \times (0, \infty)$) so there exists a measurable function $(\omega, s) \mapsto \sigma_s(\omega) \in \{-1, +1\}$ such that $|d\langle M, N\rangle_s| = \sigma_s d\langle M, N\rangle_s$. Replacing F_s with $F_s\sigma_s \mathrm{sgn}(F_s G_s)$ we see that, without loss of generality, it suffices to prove the lemma in the case where $FG \geqslant 0$ and $\langle M, N\rangle \geqslant 0$.

Since the bracket process is bilinear,

$$0 \leqslant \langle M - \alpha N\rangle = \langle M\rangle - 2\alpha\langle M, N\rangle + \alpha^2\langle N\rangle \tag{1.12}$$

for any $\alpha \in \mathbb{R}$. It follows from (1.12) that for any bounded interval $I \subseteq [0, \infty)$ and $\alpha \in \mathbb{R}$,

$$\int_s^t \mathbb{1}\{u \in I\} d\langle M\rangle_u + \alpha^2 \int_s^t \mathbb{1}\{u \in I\} d\langle N\rangle_u \geqslant 2\alpha \int_s^t \mathbb{1}\{u \in I\} d\langle M, N\rangle_u. \tag{1.13}$$

Setting $\alpha = \left(\frac{\int_s^t \mathbb{1}\{u\in I\}d\langle M\rangle_u}{\int_s^t \mathbb{1}\{u\in I\}d\langle N\rangle_u}\right)^{1/2}$ we have

$$\int_s^t \mathbb{1}\{u \in I\} d\langle M, N\rangle_u \leqslant \left(\int_s^t \mathbb{1}\{u \in I\} d\langle M\rangle_u\right)^{1/2} \left(\int_s^t \mathbb{1}\{u \in I\} d\langle N\rangle_u\right)^{1/2}.$$

Therefore, the stated result holds for simple functions (i.e. linear sums of indicator functions of bounded intervals). The usual procedure for finite variational integrals upgrades this to any $F \in \mathbb{L}^2(M)$ and $G \in \mathbb{L}^2(N)$.

To prove the second inequality, simply take expectations in (1.13) and proceed as above. ■

Lemma 1.3.6 *Let $M, N \in \mathcal{M}_0^2$ and let $F \in \mathbb{L}^2(M), G \in \mathbb{L}^2(N)$. Then for all t*

$$\left\langle \int_0^\cdot F_s dM_s, \int_0^\cdot G_s dN_s \right\rangle_t = \int_0^t F_s G_s d\langle M, N\rangle_s.$$

PROOF: First, let $F, G \in \mathbb{L}_0$. In particular, without loss of generality assume F and G have the same partition (t_i) (with $t_0 < t_1 < \ldots$) so as

$$F_s = \sum_{i=1}^{k} F_{t_i} \mathbb{1}\{s \in [t_i, t_{i+1})\}, \qquad G_s = \sum_{i=1}^{k} G_{t_i} \mathbb{1}\{s \in [t_i, t_{i+1})\}.$$

Then (in similar style to the proof of Lemma 1.2.5),

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u dN_u\right) - \int_s^t F_u G_u d\langle M, N\rangle_u \,\Big|\, \mathcal{F}_s\right] \\
&\quad = \mathbb{E}\left[\sum_i \sum_j F_{t_i} G_{t_j} (M_{t_{i+1}} - M_{t_i})(N_{t_{j+1}} - N_{t_j}) - \sum_i F_{t_i} G_{t_i} (\langle M, N\rangle_{t_{i+1}} - \langle M, N\rangle_{t_i} \,\Big|\, \mathcal{F}_s\right] \\
&\quad = 0.
\end{aligned}$$

In the above equation, with slight abuse of notation, the sums over i are over the partition $\{s, t_i, \ldots, t_{i'}, t\}$ (where $s \in (t_i, t_{i+1}]$ and $t \in (t_{i'}, t_{i'+1}]$) and similarly for j. The final line of the above follows since

$$\mathbb{E}\left[F_{t_i} G_{t_j} (M_{t_{i+1}} - M_{t_i})(N_{t_{j+1}} - N_{t_j}) \,\big|\, \mathcal{F}_{t_i \wedge t_j}\right]$$

is zero unless $i = j$, in which case

$$\begin{aligned}
&\mathbb{E}\left[F_{t_i} G_{t_i} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) - F_{t_i} G_{t_i} (\langle M, N\rangle_{t_{i+1}} - \langle M, N\rangle_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
&\quad = F_{t_i} G_{t_i} \mathbb{E}\left[(M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) - (\langle M, N\rangle_{t_{i+1}} - \langle M, N\rangle_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
&\quad = F_{t_i} G_{t_i} \mathbb{E}\left[M_{t_{i+1}} N_{t_{i+1}} - M_{t_i} N_{t_i} - (\langle M, N\rangle_{t_{i+1}} - \langle M, N\rangle_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
&\quad = 0.
\end{aligned}$$

We have thus proved the result for the case $F, G \in \mathbb{L}_0$.

Now let $F \in \mathbb{L}^2(M)$ and $G \in \mathbb{L}^2(N)$ and by Theorem 1.3.3 let $(F(n))$ and $(G(n))$ be sequences of simple processes such that $F(n) \to F$ in $\mathbb{L}^2(M)$, $G(n) \to G$ in $\mathbb{L}^2(N)$ and (1.10) holds. From above, for each n we have

$$\mathbb{E}\left[\left(\int_s^t F(n)_u dM_u\right)\left(\int_s^t G(n)_u dN_u\right) - \int_s^t F(n)_u G(n)_u d\langle M, N\rangle_u \,\Big|\, \mathcal{F}_s\right] = 0. \tag{1.14}$$

We will take the limit of the two terms in the above equation in turn, starting with the leftmost.

Using the conditional Jensen inequality followed by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}\left[\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u - G(n)_u \, dN_s\right) \,\Big|\, \mathcal{F}_s\right]\right|\right] \\
&\quad \leqslant \mathbb{E}\left[\mathbb{E}\left[\left|\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u - G(n)_u \, dN_s\right)\right| \,\Big|\, \mathcal{F}_s\right]\right] \\
&\quad = \mathbb{E}\left[\left|\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u - G(n)_u \, dN_s\right)\right|\right] \\
&\quad \leqslant \mathbb{E}\left[\left(\int_s^t F_u dM_u\right)^2\right]^{1/2} \mathbb{E}\left[\left(\int_s^t G_u - G(n)_u \, dN_u\right)^2\right]^{1/2}
\end{aligned}$$

The above equation tends to zero since, by (1.3.3) (in $L^2(\mathbb{P})$), the second term on the right hand side tends to zero as $n \to \infty$. Hence

$$\mathbb{E}\left[\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u - G(n)_u \, dN_s\right) \,\Big|\, \mathcal{F}_s\right] \to 0 \tag{1.15}$$

in $L^1(\mathbb{P})$ as $n \to \infty$. By passing to a subsequence (which we do not notate) we can assume that in fact the convergence in (1.15) is almost sure.

Similarly, we have

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}\left[\left(\int_s^t F_u - F(n)_u\,dM_u\right)\left(\int_s^t G(n)_u\,dN_s\right)\,\Big|\,\mathcal{F}_s\right]\right|\right] \\ \leqslant \mathbb{E}\left[\left(\int_s^t F(n)_u - F_u\,dM_u\right)^2\right]^{1/2}\mathbb{E}\left[\left(\int_s^t G(n)_u\,dN_u\right)^2\right]^{1/2}\end{aligned} \tag{1.16}$$

By (1.3.3) again, the first term on the right hand side of the above tends to zero as $n \to \infty$. Further, by Theorem 1.2.9 and the fact that $G(n) \to G$ in $\mathbb{L}^2(N)$ we have

$$\sup_n \mathbb{E}\left[\left(\int_s^t G(n)_u dNu\right)^2\right] = \sup_n \mathbb{E}\left[\int_s^t G(n)_s^2 d\langle N\rangle_s\right] < \infty. \tag{1.17}$$

Hence also (1.16) tends to zero as $n \to \infty$. We thus have

$$\mathbb{E}\left[\left(\int_s^t F_u - F(n)_u\,dM_u\right)\left(\int_s^t G(n)_u\,dN_s\right)\,\Big|\,\mathcal{F}_s\right] \to 0 \tag{1.18}$$

in $L^1(\mathbb{P})$ as $n \to \infty$. By passing to a further subsequence (which, again, we do not notate) we have that the convergence in (1.18) holds almost surely.

Summing (1.15) and (1.18) we obtain

$$\mathbb{E}\left[\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u dN_u\right) - \left(\int_s^t F(n)_u\right)\left(\int_s^t G(n)_u dN_u\right)\,\Big|\,\mathcal{F}_s\right] \to 0. \tag{1.19}$$

almost surely as $n \to \infty$.

We now move on to the second term of (1.14). Using the conditional Jensen inequality followed by the second part of Lemma 1.3.4 we have

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}\left[\int_s^t F_u(G_u - G(n)_u)d\langle M,N\rangle_u\,\Big|\,\mathcal{F}_s\right]\right|\right] \\ &\leqslant \mathbb{E}\left[\mathbb{E}\left[\left|\int_s^t F_u(G_u - G(n)_u)d\langle M,N\rangle_u\right|\,\Big|\,\mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[\left|\int_s^t F_u(G_u - G(n)_u)d\langle M,N\rangle_u\right|\right] \\ &\leqslant \mathbb{E}\left[\int_s^t |F_u(G_u - G(n)_u)|\,|d\langle M,N\rangle_u|\right] \\ &\leqslant \mathbb{E}\left[\left(\int_s^t F_u^2 d\langle M\rangle_u\right)^2\right]^{1/2}\mathbb{E}\left[\left(\int_s^t (G_u - G(n)_u)^2 d\langle N\rangle_u\right)^2\right]^{1/2}.\end{aligned}$$

The above equation tends to zero, because the second term on the right hand side of the above tends to zero (since $G(n) \to G$ in $\mathbb{L}^2(N)$). Hence,

$$\mathbb{E}\left[\int_s^t F_u(G_u - G(n)_u)d\langle M,N\rangle_u\,\Big|\,\mathcal{F}_s\right] \to 0 \tag{1.20}$$

in $L^1(\mathbb{P})$ as $n \to \infty$. By passing to a third subsequence (which, yet again, we do not notate) we can assume the convergence in (1.20) is almost sure.

Similarly,

$$\mathbb{E}\left[\left|\mathbb{E}\left[\int_s^t (F_u - F(n)_u)G(n)_u d\langle M,N\rangle_u\right]\right|\right]$$
$$\leqslant \mathbb{E}\left[\left(\int_s^t (F_u - F(n)_u)^2 d\langle M\rangle_u\right)^2\right]^{1/2} \mathbb{E}\left[\left(\int_s^t G(n)_u^2 d\langle N\rangle_u\right)^2\right]^{1/2}. \quad (1.21)$$

The above equation tends to zero, because the first time on the right hand side tends to zero (since $F(n) \to F$ in $\mathbb{L}^2(M)$) and the second term on the right hand side is bounded as in (1.17). Hence

$$\mathbb{E}\left[\int_s^t (F_u - F(n)_u)G(n)_u d\langle M,N\rangle_u\right] \to 0 \quad (1.22)$$

in $L^1(\mathbb{P})$ as $n \to \infty$ and by passing to a fourth subsequence (which, yet again, we do not notate) we have that the convergence in (1.22) holds almost surely.

Combining (1.20) and (1.22) we have

$$\mathbb{E}\left[\int_s^t F(n)_u G(n)_u d\langle M,N\rangle_u \,\Big|\, \mathcal{F}_s - \int_s^t F_u G_u d\langle M,N\rangle_u \,\Big|\, \mathcal{F}_s\right] \to 0. \quad (1.23)$$

Putting (1.19) and (1.23) into (1.14) we obtain

$$\mathbb{E}\left[\left(\int_s^t F_u dM_u\right)\left(\int_s^t G_u dN_u\right) - \int_s^t F_u G_u d\langle M,N\rangle_u \,\Big|\, \mathcal{F}_s\right] = 0.$$

This completes the proof. ■

1.4 Local Martingales

In this section we introduce a natural generalization of martingales, known as local martingales.

Definition 1.4.1 *A real valued stochastic process $(M_t)_{t\geqslant 0}$ is a* ***local martingale*** *if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of finite stopping times such that*

1. *$T_n \to \infty$ almost surely as $n \to \infty$.*
2. *$t \mapsto M_{t\wedge T_n}$ is a martingale (with respect to the same filtration as M) for all n.*

The sequence (T_n) is known as the localizing sequence of M.

Questions 4 and 5 on Problem Sheet 1 give examples of processes which are local martingale but not local martingales. Local martingales are typically not martingales, whereas martingales are always local martingales (for example, choose $T_n = n$). The most useful way to show that a local martingale is a martingale is the following lemma.

Lemma 1.4.2 *If M is a bounded local martingale then M is a martingale.*

PROOF: The fact that $T_n \to \infty$ almost surely implies that $M_{t\wedge T_n} \to M_t$ as $n \to \infty$. Since M is bounded we have, for all $t \geqslant 0$,

$$\mathbb{E}\left[\sup_n |M_{t\wedge T_n}|\right] < \infty \quad (1.24)$$

and thus the conditional dominated convergence theorem implies that also $\mathbb{E}[M_{t\wedge T_n}\,|\,\mathcal{F}_t] \to \mathbb{E}[M_t\,|\,\mathcal{F}_t]$ almost surely. The result follows by taking an (almost sure) limit as $n\to\infty$ of $\mathbb{E}\left[M_{(t+s)\wedge T_n}\,|\,\mathcal{F}_t\right] = M_{t\wedge T_n}$. ∎

Of course, the above proof implies that in fact (1.24) is enough to show that a local martingale is really a martingale. This apparent improvement on Lemma 1.4.2 is of no use to us; the reason (which appears below as Lemma 1.4.4) is that typically a local martingale M has localizing sequence $T_n = \inf\{t\geqslant 0\,;\,|M(t)|\geqslant n\}\wedge n$. The following lemma, in the same spirit, can also sometimes be helpful.

Lemma 1.4.3 *Let M be a non-negative local martingale. Then M is a supermartingale.*

Proof: Let (T_n) be a localizing sequence for M. Apply the conditional Fatou lemma to the left hand side of $\mathbb{E}[M_{t\wedge T_n}\,|\,\mathcal{F}_s] = M_{s\wedge T_n}$ as $n\to\infty$. ∎

A local martingale can possess many good qualities without being a martingale. For example, in Question 1 of Problem Sheet 5 we exhibit a process that is a local martingale, a supermartingale, uniformly integrable and even bounded in L^2, but which is not a martingale.

Local martingales will become very important to us; a large natural class of local martingales will appear as a consequence of Itô's formula.

Lemma 1.4.4 *Let M be a continuous local martingale with $M_0 = 0$. Then there exists a localizing sequence of stopping times (T_n) for M such that for each n, M^{T_n} is a bounded martingale and $T_n\leqslant n$.*

Proof: Let (T_n) be a localizing sequence for M and let $S_n = \inf\{t\geqslant 0\,;\,|M_t| = n\}\wedge n$. Since M is continuous and $M_0 = 0$, $S_n\uparrow\infty$ almost surely as $n\to\infty$, and hence $\tau_n = S_n\wedge T_n$ is a localizing sequence for M, with the additional property that $M_{\cdot\wedge\tau_n}\leqslant n$ and $\tau_n\leqslant n$. ∎

For local martingales, there are (as far as I know) no generalizations of the maximal inequality, L^p inequality, optional stopping theorem or martingale convergence theorem. It is often possible to treat a local martingale M as though it was a martingale, by applying 'martingale' results to $M_{\cdot\wedge T_n}$ and then letting $n\to\infty$. We will see several examples of this in future sections.

Two of the results from Section 1.1 do have direct local martingale equivalents. Firstly, it is easily seen that if $\tau<\infty$ is a stopping time and M is a local martingale then $M_{\cdot\wedge\tau}$ is a local martingale. Secondly, the bracket process has a natural extension to continuous local martingales, which we now describe.

Theorem 1.4.5 *Let M, N be continuous local martingales. Then there exists a unique continuous adapted process $\langle M,N\rangle$ with locally finite variation such that $\langle M,N\rangle_0 = 0$ and $MN-\langle M,N\rangle$ is a local martingale.*

Proof: Let τ_n be a localizing sequence for M and note that by Lemma 1.4.4 we may assume M^{τ_n} is bounded for each n and $\tau_n\leqslant n$. Therefore, by Theorem 1.1.10 the bracket process of $M_{\cdot\wedge\tau_n}$ exists; $M^2_{t\wedge\tau_n}-\langle M_{\cdot\wedge\tau_n}\rangle_t$ is a martingale. By stopping this process at τ_{n-1} we see that $M^2_{t\wedge\tau_{n-1}}-\langle M_{\cdot\wedge\tau_n}\rangle_{t\wedge\tau_{n-1}}$ is also a martingale. Therefore the uniqueness part of Theorem 1.1.10 implies that

$$\langle M_{\cdot\wedge\tau_n}\rangle_t = \langle M_{\cdot\wedge\tau_{n-1}}\rangle_t \qquad \text{for all } t\leqslant\tau_{n-1}.$$

As a result, we may define for all $t \in [0, \infty)$,

$$\langle M \rangle_t = \langle M_{\cdot \wedge \tau_n} \rangle_t \tag{1.25}$$

pathwise where n is chosen such that $t \leqslant \tau_n$. With this definition,

$$M_t^2 - \langle M \rangle_t$$

is a local martingale with localizing sequence (τ_n).

Through (1.2) we can define $\langle M, N \rangle$ for pairs of local martingales M, N. Using bilinearity and applying the above results for $M + N$ and $M - N$ we can then show that $MN - \langle M, N \rangle$ is a local martingale. With a little work it can be shown that the various properties claimed of $\langle M, N \rangle$ are inherited from Theorem 1.1.10 via (1.25); we omit these details. ■

It is easily seen that our extension of $\langle \cdot, \cdot \rangle$ preserves the bilinearity and also satisfies (1.2). This is useful to us because it means that properties of $\langle M, N \rangle$ can usually be deduced from the equivalent property for $\langle M \rangle$.

We now prepare ourselves to define Itô integrals with respect to local martingales. Crucially, we need to establish the relationship between an Itô integral stopped at time τ and an Itô integral with respect a martingale stopped a time τ.

Lemma 1.4.6 *Let M, N be local martingales and $T < \infty$ be a stopping time. Then $\langle M^T \rangle_\cdot = \langle M \rangle_{\cdot \wedge T}$ and $\langle M^T, M \rangle_{\cdot \wedge T} = \langle M \rangle_{\cdot \wedge T}$.*

PROOF: For the first statement, note that $M_t^T = M_{t \wedge T}$ and apply Theorem 1.4.5 to both M^T and M. The second statement is similar. ■

Lemma 1.4.7 *Let $M \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M)$. Let $\tau < \infty$ be a stopping time. Then*

$$\int_0^{\cdot \wedge \tau} F_s dM_s = \int_0^{\cdot} F_s dM_s^\tau.$$

PROOF: Note that the corresponding result holds for finite variational integrals. That is,

$$\int_0^{\cdot} F_s^2 d\langle M \rangle_{s \wedge \tau} = \int_0^{\cdot \wedge \tau} F_s^2 d\langle M \rangle_s. \tag{1.26}$$

By the Itô isometry (Theorem 1.2.9), followed by Lemma 1.4.6 and (1.26) we have

$$\mathbb{E}\left[\left(\int_0^t F_s dM_s^\tau\right)^2\right] = \mathbb{E}\left[\int_0^t F_s^2 d\langle M^\tau \rangle_s\right] = \mathbb{E}\left[\int_0^{t \wedge \tau} F_s^2 d\langle M \rangle_{s \wedge \tau}\right] = \mathbb{E}\left[\int_0^{t \wedge \tau} F_s^2 d\langle M \rangle_s\right]$$

and by the Itô isometry again,

$$\mathbb{E}\left[\left(\int_0^{t \wedge \tau} F_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^{t \wedge \tau} F_s^2 d\langle M \rangle_s\right].$$

Using the Optional Stopping Theorem (at time $t \wedge \tau$), followed by with Lemmas 1.3.6, 1.4.6 and finally (1.26) we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t F_s dM_s^\tau\right)\left(\int_0^{t\wedge\tau} F_s dM_s\right)\right] &= \mathbb{E}\left[\left(\int_0^{t\wedge\tau} F_s dM_s^\tau\right)\left(\int_0^{t\wedge\tau} F_s dM_s\right)\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge\tau} F_s^2 d\langle M^\tau, M\rangle_s\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge\tau} F_s^2 d\langle M\rangle_{s\wedge\tau}\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge\tau} F_s^2 d\langle M\rangle_s\right].
\end{aligned}$$

We can therefore conclude that

$$\mathbb{E}\left[\left(\int_0^t F_s dM_s^\tau - \int_0^{t\wedge\tau} F_s dM_s\right)^2\right] = 0.$$

Hence, $\int_0^t F_s dM_s^\tau$ and $\int_0^{t\wedge\tau} F_s dM_s$ are almost surely equal for a dense countable subset of t and, by continuity, almost surely equal as processes. ■

1.5 Itô Calculus II

We seek to extend the definition of the Itô integral, in the same style as our extension of the bracket process in Theorem 1.4.5.

Recall the spaces $\mathcal{M}_0^2$ and $\mathbb{L}^2(M)$. Let

$$\mathcal{M}_{0,loc} = \{M\,;\,(M_t)_{t=0}^\infty \text{ is a continuous local martingale and } M_0 = 0\}$$

and let

$$\mathbb{L}^2_{loc}(M) = \left\{F\,;\,(F_t)_{t=0}^\infty \text{ is progressively measurable and } \int_0^\infty F_s^2 d\langle M\rangle_s < \infty\right\}.$$

Let $M \in \mathcal{M}_{0,loc}$ with localizing sequence (T_n) and recall that by Lemma 1.4.4 we can choose T_n such that $M^{T_n} \in \mathcal{M}_0^2$. Let $F \in \mathbb{L}^2_{loc}(M)$. Define

$$S_n = \inf\left\{t \geqslant 0\,;\,\left|\int_0^t F_s^2 d\langle M\rangle_s\right| \geqslant n\right\}$$

and note that $S_n \uparrow \infty$. Hence

$$\tau_n = T_n \wedge S_n \wedge n \tag{1.27}$$

is a localizing sequence for M. Further,

$$\int_0^t F_s^2 d\langle M^{\tau_n}\rangle_s = \int_0^{t\wedge\tau_n} F_s^2 d\langle M\rangle_s \tag{1.28}$$

so also $F_\cdot \in \mathbb{L}^2(M^{\tau_n})$. The results of Section 1.2 will now allow us to define the process $\int_0^\cdot F_s dM_s^{\tau_n}$ pathwise. By Lemma 1.4.7 for $t \leqslant \tau_n$ we have

$$\int_0^t F_s dM_s^{\tau_{n+1}} = \int_0^{t\wedge\tau_n} F_s dM^{\tau_{n+1}} = \int_0^t F_s dM_s^{\tau_{n+1}\wedge\tau_n} = \int_0^t F_s dM_s^{\tau_n} = \int_0^{t\wedge\tau_n} F_s dM_s^{\tau_n} = \int_0^t F_s dM^{\tau_n}.$$

Thus we can define the Itô integral of F with respect to M by

$$\int_0^t F_s dM_s = \int_0^t F_s dM_s^{\tau_n} \tag{1.29}$$

where n is such that $t \leqslant \tau_n$. It is easily seen that this definition does not depend on the sequence (T_n) used to localize M.

Definition 1.5.1 *A stochastic process $(X_t)_{t \geqslant 0}$ is a continuous semimartingale if it can be written*

$$X_t = X_0 + M_t + A_t$$

where X_0 is $\mathcal{F}_0$ measurable, M is a continuous local martingale with $M_0 = 0$ and A_t is a continuous adapted process with paths of locally finite variation with $A_0 = 0$. The processes M and A are known respectively as the martingale and finite variational parts of X.

Definition 1.5.2 *Let $X = M + A$ be a semimartingale with martingale part $M \in \mathcal{M}_{0,loc}$ and finite variational part A. Let $F \in \mathbb{L}^2_{loc}(M)$. The* ***Itô integral*** *of F with respect to X is the process*

$$\int_0^\cdot F_s dX_s = \int_0^\cdot F_s dM_s + \int_0^\cdot F_s dA_s$$

where, on the right hand side, the first term is defined by (1.29) *and the second term is defined (pathwise) as a Lebesgue-Stieltjes integral.*

Definition 1.5.2 gives the most general form of the Itô integral for continuous stochastic processes. This is even a theorem! See Section IV.16 and Remark IV.34.14 of Rogers and Williams (2000) for details. We use (1.8) to extend Definition 1.5.2 and define $\int_s^t F_u dX_s = u$.

As with ordinary differential equations, it is common to drop the integral sign when making implicit definitions of stochastic processes in terms of Itô integrals. When this is done it known as a **stochastic differential equation** (or SDE). For example, the equation

$$dY_t = f(t, B_t)dt + g(t, B_t)dB_t$$

means that

$$Y_t - Y_0 = \int_0^t f(s, B_t)ds + \int_0^t g(s, B_s)dB_s$$

where the first term on the right is a Lebesgue-Stieltjes integral and the second is an Itô integral. Writing such an expression is not an assertion that there is a process solving the equation; in general there is a whole theory devoted to the existence and uniqueness of solutions to SDEs. We will not touch on that theory in this course.

We have defined the Itô integral for $\mathbb{R}$ valued processes but it extends naturally to $\mathbb{R}^d$ valued processes via componentwise operations. Of course our primary application later in the course will be in two dimensions (since $\mathbb{R}^2 \cong \mathbb{C}$). For simplicity we will continue to work in one dimension and we will move into two dimensions only when it becomes necessary to do so.

1.6 Properties of the Itô Integral II

Recall that in Section 1.3 we set out a two stage method for establishing properties of the Itô integral of Definition 1.2.6. In order to transfer such properties, where possible, onto the Itô integral of Definition 1.5.2 two additional stages are needed:

3. Pick an appropriate localizing sequence $M \in \mathcal{M}_{0,loc}$. Prove the property for the localized integral $\int_0^\cdot dM_s^{\tau_n}$ and use (1.29) to deduce the result for $\int_0^\cdot dM_s$.

4. Deduce the property for finite variational integrals and combine.

We will typically not bother with stage 4, under the tentative assumption that you are familiar with Lebesgue-Stieltjes integrals. We have already done stages 1 and 2 for some properties so in most of the lemmas below we need only carry out the third step. For example:

Lemma 1.6.1 *Let $M \in \mathcal{M}_{0,loc}$ and let $F \in \mathbb{L}^2_{loc}(M)$. Let $\tau < \infty$ be a stopping time. Then*

$$\int_0^{\cdot\wedge\tau} F_s dM_s = \int_0^\cdot F_s dM_s^\tau.$$

PROOF: We have already shown this in Lemma 1.4.7 for the case where $M \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M)$. Let (τ_n) be given by (1.27) and then by Lemma 1.4.7 we have

$$\int_0^{\cdot\wedge\tau} F_s dM_s^{\tau_n} = \int_0^\cdot F_s dM_s^{\tau_n\wedge\tau}$$

The result follows by (1.29). ■

As in Section 1.3, the easiest property to deduce is linearity in terms of the integrand.

Lemma 1.6.2 *Let $M \in \mathcal{M}_{0,loc}$ and $F, G \in \mathbb{L}^2_{loc}(M)$. Then*

$$\int_0^\cdot F_s + G_s\, dM_s = \int_0^\cdot F_s dM_s + G_s dM_s$$

PROOF: Let (T_n) be a localizing sequence for M and recall that by Lemma 1.4.4 we can choose T_n such that $M^{T_n} \in \mathcal{M}_0^2$. Define

$$S_n = \inf\left\{t \geqslant 0\,;\, \left|\int_0^t F_s^2 d\langle M\rangle_s\right| \geqslant n\right\}$$
$$R_n = \inf\left\{t \geqslant 0\,;\, \left|\int_0^t G_s^2 d\langle M\rangle_s\right| \geqslant n\right\}$$

and set

$$\tau_n = T_s \wedge S_n \wedge R_n \wedge n.$$

Then (τ_n) is a localizing sequence for M and $F, G \in \mathbb{L}^2(M^{\tau_n})$ for all n.

Hence, by Lemma 1.3.1 we have

$$\int_0^\cdot F_s + G_s\, dM_s^{\tau_n} = \int_0^\cdot F_s dM_s^{\tau_n} + \int_0^\cdot G_s dM_s^{\tau_n}.$$

The result the follows by (1.29). ■

Of course, we require the extension of Lemma 1.3.6.

Lemma 1.6.3 *Let $M, N \in \mathcal{M}_{0,loc}$ and let $F \in \mathbb{L}^2_{loc}(M), G \in \mathbb{L}^2_{loc}(N)$. Then for all t*

$$\left\langle \int_0^\cdot F_s dM_s, \int_0^\cdot G_s dN_s \right\rangle_t = \int_0^t F_s G_s d\langle M, N\rangle_s.$$

PROOF: Recall that in Lemma 1.3.6 we have proved this result for the case where $M, N \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M), G \in \mathbb{L}^2(N)$.

Let $M, N \in \mathcal{M}_{0,loc}$ and $F \in \mathbb{L}^2_{loc}(M), G \in \mathbb{L}^2_{loc}(N)$. Let (τ^F_n) be as in (1.27), let (τ^G_n) be the equivalent sequence for G and set $\tau_n = \tau^F_n \wedge \tau^G_n$. Then (τ_n) is a localizing sequence for both M and N and $\left|\int_0^t F_s d\langle M\rangle_s\right|, \left|\int_0^{\tau_n} G_s d\langle N\rangle_s\right|$ are both bounded above by n. Hence, for each n we have $M^{\tau_n}, N^{\tau_n} \in \mathcal{M}_0^2$ and $F \in \mathbb{L}^2(M^{\tau_n}), G \in \mathbb{L}^2(N^{\tau_n})$. By Lemma 1.3.6 for all s, t, n we thus have

$$\left\langle \int_0^\cdot F_s dM^{\tau_n}_s, \int_0^\cdot G_s dN^{\tau_n}_s \right\rangle_\cdot = \int_0^\cdot F_u G_s \langle M^{\tau_n}, N^{\tau_n}\rangle_s$$

By applying Lemmas 1.6.1 and 1.4.6 to the left hand side and Lemma 1.4.6 to the right hand side we obtain

$$\left\langle \int_0^\cdot F_s dM_s, \int_0^\cdot G_s dN_s \right\rangle_{\cdot\wedge\tau_n} = \int_0^{\cdot\wedge\tau_n} F_u G_s \langle M, N\rangle_s.$$

Since $\tau_n \to \infty$ almost surely as $n \to \infty$, the result follows. ■

A notable omission in our construction so far is that Brownian motion $(B_t)_{t=0}^\infty$ is a square integrable martingale but not an L^2 bounded martingale; $B \notin \mathcal{M}_0^2$. Consequently our first version of the Itô isometry (Theorem 1.2.9) did not apply to Brownian motion. However, B is most certainly a local martingale and, for each $T > 0$, $(B_t)_{t=0}^T$ *is* an L^2 bounded martingale. In this case and in others like it, an analogue of Theorem 1.2.9 holds.

Corollary 1.6.4 (Itô isometry, II) *Suppose that $M \in \mathcal{M}_{0,loc}$ and and suppose $F \in \mathbb{L}^2_{loc}(M)$. Suppose that $\mathbb{E}\left[M_T^2\right] < \infty$ and $\int_0^T \mathbb{E}\left[F_s^2\right] d\langle M\rangle_s < \infty$ for some $T > 0$. Then*

$$\mathbb{E}\left[\left(\int_0^t F_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t F_s^2 d\langle M\rangle_s\right] \tag{1.30}$$

for all $t \in [0, T]$. Further, if $N \in \mathcal{M}_{0,loc}$ and $G \in \mathbb{L}^2_{loc}(N)$, with $\mathbb{E}[N_T^2] < \infty$ and $\int_0^T \mathbb{E}\left[G_s^2\right] d\langle N\rangle_s < \infty$ then

$$\mathbb{E}\left[\left(\int_0^t F_s dM_s\right)\left(\int_0^t G_s dN_s\right)\right] = \mathbb{E}\left[\int_0^t F_s G_s d\langle M, N\rangle_s\right]. \tag{1.31}$$

for all $t \in [0, T]$.

PROOF: The first statement is a special case of the second. The second statement follows from Lemma 1.3.6 and appropriate localization. ■

Note in particular that if M is Brownian motion then the above theorem holds for all $T > 0$. This happens in other important cases too.

Remark 1.6.5 *We could have started out defining the Itô integral over intervals of time $[0, T]$ where $T < \infty$, instead of $[0, \infty)$. Had we done so we could have obtained Theorem 1.6.4 much sooner, but at the cost of subsequently having to fiddle around with two (instead of just one) objects that were tractable only in a local sense.*

We need one final 'elementary' property of the Itô integral, namely the result of taking one Itô integral with respect to another. Formally, the result is the same as for Lebesgue-Stieltjes integrals.

Lemma 1.6.6 *Suppose that $M \in \mathcal{M}_{0,loc}$. Suppose that for each $t \geqslant 0$, the process $(s,\omega) \mapsto F_t(\omega)$ for $s \in [0,t], \omega \in \Omega$ is measurable with respect to $\mathscr{B}[0,t] \times \mathcal{F}_t$. Suppose that $G \in \mathbb{L}^2_{loc}(M)$ and $FG \in \mathbb{L}^2_{loc}(M)$. Then $F \in \mathbb{L}_{loc}(\int_0^\cdot G_s dM_s)$ and for all t*

$$\int_0^t F_s d\left(\int_0^s G_u dM_u\right) = \int_0^t F_s G_s dM_s.$$

Proof: See Question 2 on Problem Sheet 2. ■

1.7 Itô's Formula

We need only one further tool in order to calculate fluently with stochastic integrals, namely Itô's formula. In the spirit of the chain rule for finite variational integrals, Itô's formula allows us to to express $f(s, F_s)$ in terms of integrals of the derivatives of f.

Definition 1.7.1 *Let $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$. We will often write $f(t, x_1, x_2, \dots, x_d) = f(t,x)$ where $x = (x_i)_1^d$. The t coordinate is referred to as the* ***time coordinate*** *and the x coordinate as the* ***spatial coordinate****.*

When writing Itô's formula in dimension $d > 1$ it is normal to use the subscript to indicate dimension and place the time coordinate as the primary argument. For example, we would usually write $X = (X_i)_{i=1}^d$ for an $\mathbb{R}^d$ valued stochastic process with i^{th} coordinate projection X_i where $X_i = (X_i(t))_{t=0}^\infty$.

Of course this clashes with our usual notation and we hope no confusion occurs. Except for in this section, the only dimension that interests us will be $\mathbb{R}^2 \sim \mathbb{C}$, in which we will write stochastic processes as $Z_t = X_t + iY_t$ (for real X_t, Y_t), thus avoiding any potential confusion.

Theorem 1.7.2 (Itô's formula) *Let $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ and suppose that the derivatives f_t and $f_{x_i x_j}$ exist and are continuous (for all $1 \leqslant i,j \leqslant d$). For $i = 1, \dots, d$, suppose that $X_i(\cdot) = X_i(0) + M_i(\cdot) + A_i(\cdot)$ is a continuous semimartingale with martingale part M_i and finite variational part A_i. Then, for all t,*

$$\begin{aligned} f(t, X(t)) - f(0, X(0)) = &\int_0^t f_t(s, X(s))ds \\ &+ \sum_{i=1}^d \int_0^t f_{x_i}(s, X(s))dM_i(s) + \sum_{i=1}^d \int_0^t f_{x_i}(s, X(s))dA_i(s) \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t f_{x_i x_j}(s, X(s))d\langle M_i, M_j\rangle_s. \end{aligned} \tag{1.32}$$

Sketch of Proof: Proving Itô's formula is a serious piece of work and here we restrict ourselves to a non-rigorous outline of one method of proof.

First, we note that Itô's formula holds for linear functions $f(t,x) = a_0 + t_0 t + \sum_{i=1}^d a_i x_i$. This follows from nothing more than the linearity of the Itô integral and the formula

$$\int_0^t dX_i(s) = X_i(t) - X_i(s),$$

which is itself easy to prove using Theorem 1.3.3.

Secondly, we look to prove Itô's formula in the special case $f(t,x) = x^2$, where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$. To this end, let us briefly write $X_t = M_t + A_t$ where X_t is an $\mathbb{R}$ valued continuous semimartingale. Let $0 = t_0 < t_1 < \ldots < t_n = t$ and note that

$$X_t^2 - X_0^2 = \sum_{j=1}^{n}(X_{t_j}^2 - X_{t_{j-1}}^2) = \sum_{j=1}^{n} 2X_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) + \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})^2. \tag{1.33}$$

For the first term on the right hand side we have

$$\sum_{j=1}^{n} 2X_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) = \sum_{j=1}^{n} 2X_{t_{j-1}} \int_{t_{j-1}}^{t_j} dX_s = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} 2X_{t_{j-1}} dX_s = \int_s^t \sum_{j=1}^{n} \mathbb{1}\{s \in [t_{i-1}, t_j)\} f'(X_{t_{j-1}}) dX_s$$

and, as $\sup_j |t_j - t_{j-1}| \to 0$ we have (at least, heuristically) that

$$\int_s^t \sum_{j=1}^{n} \mathbb{1}\{s \in [t_{i-1}, t_j)\} f'(X_{t_{j-1}}) dX_s \to \int_s^t f'(X_s) dX_s.$$

For the second term on the right hand side of (1.33) we have

$$\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2 = \sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2 + 2\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})(A_{t_i} - A_{t_{i-1}}) + \sum_{i=1}^{n}(A_{t_i} - A_{t_{i-1}}). \tag{1.34}$$

The rightmost term on the right hand side of the above equation tends to zero as $\sup_j |t_j - t_{j-1}| \to 0$ because A has finite variation. By (1.1), the leftmost term on the right hand side tends to $\langle M \rangle_t$. The middle term thus also tends to zero by the Cauchy-Schwarz inequality. We thus obtain

$$X_t^2 - X_0^2 = \int_0^t 2X_s dX_s + \langle M \rangle_t,$$

and the case $f(x) = x^2$ is completed by noting $f_{xx} = 2$ and

$$\langle M \rangle_s = \int_0^t d\langle M \rangle_s = \frac{1}{2}\int_0^t 2d\langle M \rangle_s$$

which completes the case $f(x) = x^2$.

From the two special cases computed above and Taylor's theorem, the full version of Itô's formula can be deduced. Let us briefly outline how this is done. Firstly, we use (1.2) and the polarization identity $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$ to upgrade the case $f(x) = x^2$ into the case where $f(x_1, \ldots, x_d)$ is a quadratic form. We then apply (the multivariate) Taylor's theorem to $f(t,x)$, writing out both the first and second order terms. Various applications of localization, finite variation, the martingale property and the cases of Itô's formula that we already know come together and allow us to match up the non-vanishing terms to (1.32) and control the remaining error terms. ■

Remark 1.7.3 *Itôs formula is often abbreviated to read*

$$df(t,X(t)) = f_t(t,X(t))dt + \sum_{i=1}^{d} f_{x_i}(t,X(t))dX_i(t) + \frac{1}{2}\sum_{i,j=1}^{d} f_{x_i x_j}(t,X(t))dX_i(t)dX_j(t) \tag{1.35}$$

with the convention that $dX_i(t) = dM_i(t) + dA_i(t)$ *and the 'multiplication table'*

	$dA_j(t)$	$dM_j(t)$
$dA_i(t)$	0	0
$dM_i(t)$	0	$d\langle M_i, M_j\rangle_t$

There is more than just an abuse of notation involved here. Let us attempt to wave our hands and describe what is going on. In the world of finite variational functions, $\int_0^{\cdot} \ldots dA_i(t)dA_j(t)$ is formally the zero operator, since the multiplication of two $d(\cdot)s$ causes a lower order term that a single Lebesgue-Stieltjes $\int$ is unable to pick up. However, (1.1) *means that the paths of continuous (local) martingales oscillate in such a way as creates terms $\int_0^{\cdot} \ldots dM_i(t)dM_j(t)$ where the Itô $\int$ does pick up something non-zero. The cross terms $dA_i(t)dM_j(t)$ become zero because M is a martingale.*

See (1.34) *and the paragraph immediately below it in our sketch proof of Itô formula to see this working in practice.*

Corollary 1.7.4 *Let $f : [0,\infty)\mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable and let X be an $\mathbb{R}^d$ valued continuous semimartingale. Then $t \mapsto f(t, X_t)$ is an $\mathbb{R}$ valued continuous semimartingale.*

Chapter 2

Complex Brownian Motion

This chapter begins by transferring much of the real Itô calculus from the preceding chapter into $\mathbb{C}$. We will then use this machinery to build up a picture of the behaviour of complex Brownian paths and prove the conformal invariance of Brownian motion.

Let us recall the definition of complex Brownian motion from (0.1).

Definition 2.0.1 *A complex valued stochastic process $Z = X + iY$ is a* ***complex Brownian motion*** *if X and Y are independent real Brownian motions.*

Recall also that an $\mathbb{R}^d$ valued process $X = (X_i)_1^d$ is a Brownian motion if and only if X is a continuous process with independent increments with $X_t - X_s \sim \mathcal{N}(0, t-s)$. It follows from this definition that $\{X_i\,;\, i \leqslant d\}$ is a set of mutually independent real Brownian motions. Consequently, the following statements are equivalent:

- $X + iY$ is a complex Brownian motion.
- (X, Y) is a Brownian motion in $\mathbb{R}^2$.
- X and Y are independent real Brownian motions.

Complex Brownian motion is a truly extraordinary object and from here on becomes the centrepiece of the course. Dimension 2 (i.e. $\mathbb{C} \cong \mathbb{R}^2$) is sometimes referred to as the critical dimension for (Euclidean) Brownian motion. The reason for this terminology is as follows.

In $\mathbb{R}$, Brownian motion is strongly recurrent; not only does it hit every $x \in \mathbb{R}^d$ almost surely but it also returns to each $x \in \mathbb{R}^d$ infinitely many times. In $\mathbb{R}^3$ and higher dimensions, Brownian motion is transient. In fact, for each x at which it does not start there is positive probability that, for some $\epsilon > 0$, Brownian motion (run for all time) does not even hit $\mathcal{B}(x, \epsilon)$.

In $\mathbb{C} \cong \mathbb{R}^2$, Brownian motion strikes a delicate balance between transience and recurrence. As we will see in this chapter, complex Brownian motion almost surely does not hit any given deterministic point $x \in \mathbb{R}^d$ (unless it starts there) but, in spite of this, the closure of the Brownian path is almost surely the entire complex plane. This unusual mixture underpins the usefulness of Brownian motion as a tool in complex analysis; in some sense Brownian motion visits all of the complex plane but in another sense it visits none of it.

Remark 2.0.2 *Trivially, if $z \in \mathbb{C}$ and $Z.$ is a complex Brownian motion with $Z_0 = 0$ then $W_t = z + Z.$ is a complex Brownian motion with $W_0 = z$. Further, given Z_0 Definition 2.0.1 uniquely specifies the distribution of complex Brownian motion (as a path valued process). With* *this in mind, for simplicity we will often state results with the assumption that $Z_0 = 0$, even if it is not really needed.*

2.1 Harmonic Functions

We note that $\mathbb{C} = \{x+iy\,;\, x,y \in \mathbb{R}\}$ is in natural bijective correspondence with $\mathbb{R}^2 = \{(x,y)\,;\, x,y \in \mathbb{R}\}$ through

$$x + iy \leftrightarrow (x, y). \tag{2.1}$$

We equip both $\mathbb{R}^2$ and $\mathbb{C}$ with their Euclidean norms and note that (2.1) is an isometric isomorphism. This correspondence will allow us to transfer seamlessly between $\mathbb{C}$ and $\mathbb{R}^2$. Note in particular that a function $f : \mathbb{C} \to \mathbb{C}$ can be written as

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{2.2}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}^2$, $z = x + iy$ and $x, y \in R$. In what follows, whenever we use the symbols f, u, v, z, x, y we implicitly cast them in the form defined by (2.2).

Recall that a function $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$ is open, is **analytic** if the limit

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \tag{2.3}$$

exists for all $z \in D$. A cursory glance at the above formula might suggest that analytic functions are the complex equivalent of what, in real analysis, would be simply called differentiable functions. In fact, it can be shown that analytic functions are smooth - they are infinitely differentiable in the sense defined in (2.3).

We define

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

These operators (which are pronouced *dee* and *dee-bar*) will appear in the complex Itô formula and, in the following lemma, offer an especially concise way of stating the Cauchy-Riemann equations.

Remark 2.1.1 *We will use several notations for derivatives interchangably; f', f_z and $f^{(1)}$ all refer to the (complex) derivative of f, whereas f_x, and $\frac{\partial f}{\partial x}$ refer to the derivative taken in the direction of the x coordinate.*

Lemma 2.1.2 (Cauchy-Riemann equations) *Let D be a domain and let $f : D \to \mathbb{C}$. Then f is analytic if and only if $\bar{\partial} f = 0$ and, in this case, $f' = \partial f$.*

PROOF: Recall the Cauchy-Riemann equations: $f = u + iv$ is analytic if and only if $u_x = v_y$ and $u_y = -v_x$. Note that

$$2\bar{\partial} f = u_x + iv_x + i(u_y + iv_y) = u_x - v_y + i(v_x + u_y).$$

Hence $\bar{\partial} f = 0$ holds if and only if the Cauchy-Riemann equations hold, which proves the first statement. For the second, note that if f is analytic then

$$2\partial f = u_x + iv_x - i(u_y + iv_y) = 2(u_x + iv_x) = 2f_x = 2f_z,$$

hence $f' = \partial f$. ■

Definition 2.1.3 *We say that a twice differentiable function f (between open subsets of $\mathbb{R}^2$ or $\mathbb{C}$) is harmonic if*

$$\Delta f = 0,$$

where $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

The Laplacian Δ can be written in terms of the ∂ and $\bar{\partial}$ operators. Suppose D is a domain and $f : D \to \mathbb{C}$ is twice differentiable (in the complex sense), then a short calculation shows that

$$\partial\bar{\partial} f = \bar{\partial}\partial f = \frac{1}{4}\Delta f. \tag{2.4}$$

The precise relationship between analytic and harmonic functions is described in the following two lemmas. Note in particular that the following lemma implies that all analytic functions are harmonic.

Lemma 2.1.4 *If $f = u + iv$ is analytic then u and v are harmonic.*

PROOF: If f is analytic then by Lemma 2.1.2 $\partial f = f'$, which is analytic, hence $\bar{\partial}\partial f = \Delta f = 0$. ■

Lemma 2.1.5 *Suppose that D is a simply connected domain. Let $z_0 \in D$ and suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function in D. Then there exists a harmonic function $v : \mathbb{R}^2 \to \mathbb{R}$ such that $f = u + iv$ is analytic in D. Further, if we specify $f(z_0)$ then v is unique.*

PROOF: This is a standard result in complex analysis and we omit the proof. ■

2.2 Martingales and Itô calculus (in $\mathbb{C}$)

In keeping with the $z = x + iy$ convention introduced in the previous section, we will often denote a complex valued stochastic processes by

$$Z_t = X_t + iY_t$$

where X and Y are real valued stochastic processes. The definition of a martingale extends naturally from $\mathbb{R}$ into $\mathbb{R}^d$ and $\mathbb{C}$.

Definition 2.2.1 *An $\mathbb{R}^d$ valued stochastic process $X = (X_j)_{j=1}^d$ is a (local) martingale if, for each j, X_j is a real valued martingale, with respect to the same filtration as X. A complex valued stochastic process $Z = X + iY$ is a **complex (local) martingale** if the $\mathbb{R}^2$ valued process (X, Y) is a martingale.*

The bracket process extends naturally to $\mathbb{C}$ valued square integrable martingales and local martingales, using bilinearity to expand out into real/imaginary parts and then using the real definitions. To be precise, if $Z = X + iY$ and $W = U + iV$ are two complex valued local martingales split respectively into real and imaginary parts then

$$\begin{aligned}\langle Z, W\rangle_t &= \langle X + iY, U + iV\rangle_t \\ &= \langle X, U\rangle_t - \langle Y, V\rangle_t + i\,\langle X, V\rangle_t + i\,\langle Y, U\rangle_t, \\ \langle \alpha Z, \beta W\rangle &= \alpha\beta\,\langle Z, W\rangle\end{aligned}$$

where $\alpha, \beta \in \mathbb{C}$.

Remark 2.2.2 *With this definition, $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on both $\mathbb{R}$ and $\mathbb{C}$. As always, when extending a bilinear form from $\mathbb{R}$ to $\mathbb{C}$ there is a choice between bilinearity and sesquilinearity in $\mathbb{C}$. In this particular case there is no established convention and some other texts use a sesquilinear $\langle \cdot, \cdot \rangle$ (in which case $\langle \alpha Z, \beta W\rangle = \alpha\overline{\beta}\langle Z, W\rangle$).*

Lemma 2.2.3 *If Z and W are square integrable complex $(\mathcal{F}_t)$ martingales then $ZW - \langle Z, W\rangle$ is a complex $(\mathcal{F}_t)$ martingale. Similarly, if Z and W are complex local $(\mathcal{F}_t)$ martingales then $ZW - \langle Z, W\rangle$ is a complex local $(\mathcal{F}_t)$ martingale.*

In both cases, the bracket process $\langle Z, W\rangle$ is the unique continuous $(\mathcal{F}_t)$ adapted process of bounded variation with this property.

PROOF: This is easily deduced from bilinearity and the corresponding results for real martingales (Theorems 1.1.10 and 1.4.5). ■

Recall equation (1.1), which (after setting $M = N$) states that

$$\langle M\rangle_t = \lim_{\sup_i |t_{i+1}-t_i| \to 0} \sum_{i=1}^{n} (M_{t_{i+1}} - M_{t_i})^2 \tag{2.5}$$

In words, $t \mapsto \langle M\rangle_t$ counts the amount of 2-variation that the path $t \mapsto M_t$ experienced during the interval $[0, t]$. Consequently, $\langle M\rangle$ acts as an internal clock for M, not counting time or distance travelled but counting a very particular kind of oscillation.

Remark 2.2.4 *If a real local martingale M has paths of finite variation and is such that $M_0 = 0$, then $M = 0$ for all time. See Theorem IV.30.4 in Rogers and Williams (2000).*

We would like to extend this functionality of the real bracket process into the complex world. However, a process in $\mathbb{C}$ can move in many more directions than a process in $\mathbb{R}$, so oscillations can be significantly more complicated in their nature. Consequently, in $\mathbb{C}$ it turns out that in order to sensibly condense the information into a single number we should further restrict the class of processes that we are interested in.

Definition 2.2.5 *A complex (local) martingale $Z = X + iY$ is said to be **conformal** if $\langle Z, Z\rangle = 0$.*

There is a subtlety to this definition which will become visible to us primary through 'lucky' cancellations in formulas involving conformal martingales. It is not easy to capture the root cause, but let us make an attempt at doing so.

We have commented above that the bracket process in $\mathbb{R}$ acts as an internal clock. Since $M^2 - \langle M\rangle$ is a martingale whenever M is a square integrable martingale, the bracket process is intimately connected to the behaviour of M; anything we know about the bracket processes gives information about M. Conformality of $Z = X + iY$ means that

$$\langle Z, Z\rangle = \langle X\rangle - \langle Y\rangle + 2i\langle X, Y\rangle = 0$$

which, comparing real and imaginary parts, implies that $\langle X\rangle = \langle Y\rangle$ (pathwise!) and $\langle X, Y\rangle = 0$. Therefore, a conformal (local) martingale Z has real and imaginary parts with their internal clocks $\langle X\rangle$ and $\langle Y\rangle$ running at exactly equal rate. Further, $\langle X,Y\rangle = 0$ implies that in some sense X and Y evolve without becoming too dependent on each other (but they need not be actually independent).

Note that if $Z = X + iY$ is a conformal local martingale then

$$\begin{aligned}\langle Z,\bar{Z}\rangle &= \langle X\rangle + \langle Y\rangle - i\langle X,Y\rangle + i\langle Y,X\rangle \\ &= 2\langle X\rangle.\end{aligned}$$

Consequently, for conformal local martingales the process $\langle Z,\bar{Z}\rangle$ plays the same role that $\langle M\rangle$ played for real local martingales.

Remark 2.2.6 *Because of the potential for confusion between $\langle Z,Z\rangle$ and $\langle Z,\bar{Z}\rangle$ we do not use the notation $\langle Z\rangle$ for complex martingales.*

Lemma 2.2.7 *Let Z be a complex martingale. Then Z is conformal if and only if $\bar{Z}$ is conformal.*

PROOF: This is immediate since $\langle \bar{Z},\bar{Z}\rangle = \langle X\rangle - \langle Y\rangle - 2i\,\langle X,Y\rangle$. ■

Naturally, we have the following lemma.

Lemma 2.2.8 *Complex Brownian motion is a conformal martingale.*

PROOF: Let us write $Z = X + iY$ for independent real Brownian motions X and Y and note that $\langle Z\rangle = \langle X\rangle - \langle Y\rangle + 2i\,\langle X,Y\rangle$. Since X and Y are independent we have $\langle X,Y\rangle = 0$ and since X and Y are real Brownian motions we have $\langle X\rangle_t = \langle Y\rangle_t = t$. The result follows. ■

We have already mentioned that Itô calculus extends naturally to $\mathbb{R}^d$ via componentwise operations. By this we mean that if $Z_t = X_t + iY_t$ is a complex stochastic process and $W_t = U_t + iV_t$ is a complex local martingale then

$$\int_0^t Z_t dW_t = \int_0^t X_t dU_t - \int_0^t Y_t dV_t + i\left(\int_0^t X_t dV_t + \int_0^t Y_t dU_t\right)$$

providing X, Y, U, V are such that all terms of the right hand side can be defined as real Itô integrals according to Definition 1.5.2.

Many properties of the Itô integral carry over naturally from $\mathbb{R}$ to $\mathbb{C}$; for example linearity in both the integrand and integrator are automatic. For clarity we record here precise complex versions of some other properties.

Let M and N be continuous complex local martingales and let F and G be complex valued processes. For each of the formulas below, it is required that M, N, F, G satisfy the conditions necessary for the Itô integrals which appear in the corresponding formula to exist (i.e. the real/imaginary parts of these processes are such that the corresponding real valued integrals exist, as defined in Chapter 1, over the same filtered space). Then for $0 \leqslant u \leqslant t \leqslant \infty$, $\alpha \in \mathbb{C}$

and stopping times $\tau < \infty$

$$
\begin{aligned}
\alpha \int_0^t F_s dM_s &= \int_0^t \alpha F_s dM_s \\
\int_0^t F_s dM_s + \int_0^t G_s dM_s &= \int_0^t F_s + G_s dM_s \\
\int_0^u F_s dM_s + \int_u^t F_s dM_s &= \int_0^t F_s dM_s \\
\int_0^t F_s d\left(\int_0^s G_s dM_s\right) &= \int_0^t F_s G_s dM_s \\
\left\langle \int_0^\cdot F_s dM_s, \int_0^\cdot G_s dN_s \right\rangle_t &= \int_0^t F_s G_s d\langle M, N\rangle_s \\
\int_0^{t\wedge\tau} F_s dM_s &= \int_0^t F_s dM_{s\wedge\tau}. \\
\overline{\int_0^t F_s dM_s} &= \int_0^t \bar{F}_s d\bar{M}_s
\end{aligned}
$$

In each case, the formula can be deduced from its corresponding real equivalent, using the component wise definitions of integrals taking values in $\mathbb{R}^2 \cong \mathbb{C}$. From this point onwards we will use the above properties of the Itô integral without comment.

Recall that Itô's formula gave us an expression for $f(X_t)$ in terms of the derivatives of f and of the process $(X_s)_{s<t}$. In $\mathbb{C} \cong \mathbb{R}^2$ the Ito formula for $f(Z)$ has many potential simplifications that can be caused by properties of both f and X.

The 'full' Itô formula for $f(t, Z_t)$ can be easily found by rewriting Theorem 1.7.2 in notation for $\mathbb{C}$ rather than $\mathbb{R}^2$. In fact, we will only need the special case where $f(t, Z_t) = f(Z_t)$.

Theorem 2.2.9 (Itô's formula in $\mathbb{C}$) *Suppose that Z is a complex continuous local martingale and that $f : \mathbb{C} \to \mathbb{C}$ be twice continuously differentiable. Then*

$$
\begin{aligned}
f(Z_t) - f(Z_0) = &\int_0^t \partial f(Z_s) dZ_s + \int_0^t \bar{\partial} f(Z_s) d\bar{Z}_s \\
&+ \int_0^t \partial\partial f(Z_s) d\langle Z, Z\rangle_s + \int_0^t \bar{\partial}\bar{\partial} f(Z_s) d\langle \bar{Z}, \bar{Z}\rangle_s \\
&+ 2\int_0^t \partial\bar{\partial} f(Z_s) d\langle Z, \bar{Z}\rangle_s.
\end{aligned}
$$

Proof of the above theorem is Question 1 on Problem Sheet 3 and involves nothing more than rewriting the real Itô formula with componentwise operations in $\mathbb{C} \cong \mathbb{R}^2$. The potential simplifications to Theorem 2.2.9 are the following.

- If Z is a conformal local martingale then by Lemma 2.2.7 $\langle Z, Z\rangle = \langle \bar{Z}, \bar{Z}\rangle = 0$. Consequently the second line vanishes.
- If f is harmonic then by (2.4) we have $\partial\bar{\partial} f = 0$. Consequently the third line vanishes.
- if f is analytic then $\partial f = f'$ and $\bar{\partial} f = 0$ by Lemma 2.1.2. In this case by Lemma 2.1.4 f is also harmonic, so we have $f(Z_t) - f(Z_0) = \int_0^t f'(Z_s) dZ_s + \int_0^t f''(Z_s) d\langle Z, Z\rangle_s$.

The case that we most often require is when *all* of the above simplifications occur.

Corollary 2.2.10 *Suppose that Z is a conformal complex continuous local martingale taking values in a domain $D \subseteq \mathbb{C}$ and suppose that f is analytic on D. Then*

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)dZ_s. \tag{2.6}$$

Further, $f(Z_t)$ is a conformal complex continuous local martingale and

$$\langle f(Z.), \overline{f(Z.)} \rangle_t = \int_0^t |f'(Z_s)|^2 ds.$$

PROOF: Equation (2.6) follows from Theorem 2.2.9 and our comments above about its simplifications. The second statement follows by using (2.6) to calculate $\langle Z, Z \rangle$ and $\langle Z, \bar{Z} \rangle$. ■

No comment on the fortunate nature of (2.6) should be required. In fact, with a little more work we can also prove the 'chain and product rules'.

Lemma 2.2.11 *Let Z and W be conformal local martingales. Then*

$$Z_t W_t - Z_0 W_0 = \int_0^t Z_s dW_s + \int_0^t W_s dZ_s$$

PROOF: This follows from the real Itô formula and the componentwise definition of complex Itô integrals. ■

Lemma 2.2.12 *Let $f : D \to \mathbb{C}$ and $g : f(D) \to \mathbb{C}$ be analytic and let Z be a conformal local martingale. Then*

$$f(g(Z_t)) - f(g(Z_0)) = \int_0^t f'(g(Z_t))dg(Z_t).$$

PROOF: By Corollary 2.2.10, $g(Y_t)$ is a conformal local martingale. A further application of Corollary 2.2.10, to $f(Y_t)$ where $Y_t = g(Z_t)$, yields the stated result. ■

2.3 Time Change

We are now well equipped to examine complex Brownian motion as a process in its own right. In this section we will prove two well known results; Lévy's characterization of Brownian motion and the time change which connects local martingales to Brownian motion.

Lemma 2.3.1 *Let $M = (M_j)_{j=1}^d$ be a continuous $\mathbb{R}^d$ valued process adapted to the filtration $(\mathcal{F}_t)$. Then M is a Brownian motion if and only if*

$$\mathbb{E}\left[\exp\left(i\xi.(M_t - M_s)\right) \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\xi|^2(t-s)\right) \tag{2.7}$$

for all $s \leqslant t$ and $\xi \in \mathbb{R}^d$.

Remark 2.3.2 *Note that the dot in $\xi.(M_t - M_s)$ denotes the dot product of vectors in $\mathbb{R}^d$. As we will see in the proof, equation (2.7) implies that the increment $M_t - M_s$ has the correct characteristic function and is independent of $(M_u)_{u<s}$.*

Proof: To see the forwards implication, simply use the Markov property of Brownian motion at time s and then calculate the characteristic function of the normal distribution. Note that there is a slightly technicality here; the filtration $(\mathcal{F}_t)$ may be larger than the generated filtration of M, but the Markov property is still valid.

For the reverse implication, note that (2.7) implies that $M_t - M_s$ has the characteristic function of a normal distribution with mean 0 and variance $t-s$, which it must therefore be. It remains to show that, given $t_1 < t_2 < \ldots < t_n$, $(M_{t_1}, M_{t_2} - M_{t_1}, \ldots, M_{t_n} - M_{t_{n-1}})$ are a set of independent random variables. To see this, fix $(\xi^l)_{l=1}^n \subseteq \mathbb{R}^d$, set $X_0 = 0$ and use (2.7) iteratively to see that

$$\begin{aligned}
\mathbb{E}\left[\prod_{l=1}^n \exp\left(i\xi^l.(M_{t_l} - M_{t_{l-1}}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\prod_{l=1}^n \exp\left(i\xi^l.(M_{t_l} - M_{t_{l-1}}\right) \Big| \mathcal{F}_{t_{l-1}}\right]\right] \\
&= \mathbb{E}\left[\prod_{l=1}^{n-1} \exp\left(i\xi^l.(M_{t_l} - M_{t_{l-1}}\right) \mathbb{E}\left[\exp\left(i\xi^n.(M_{t_n} - M_{t_{n-1}}\right) \mid \mathcal{F}_{t_{n-1}}\right]\right] \\
&= \exp\left(-\frac{1}{2}|\xi^n|^2(t_n - t_{n-1})\right) \mathbb{E}\left[\prod_{l=1}^{n-1} \exp\left(i\xi^l(M_{t_l} - M_{t_{l-1}}\right)\right] \\
&= \prod_{l=1}^n \exp\left(-\frac{1}{2}|\xi^l|^2(t_l - t_{l-1})\right) \\
&= \prod_{l=1}^n \mathbb{E}\left[\exp\left(i\xi^l.(M_{t_l} - M_{t_{l-1}}\right)\right].
\end{aligned}$$

This implies (using standard properties of characteristic functions) that the increments $M_{t_l} - M_{t_{l-1}}$ are mutually independent. ∎

Theorem 2.3.3 (Lévy characterization of Brownian motion) *Let $M = (M_j)_{j=1}^d$ be an $\mathbb{R}^d$ valued process with continuous paths. Suppose that M is adapted to the filtration $(\mathcal{F}_t)$. Then M is a Brownian motion if and only if all the following conditions hold:*

1. *Each process $t \mapsto M_j(t)$ is a local $(\mathcal{F}_t)$-martingale in $\mathbb{R}$.*
2. *For any $i, j \leqslant d$, $\langle M_j(t), M_k(t)\rangle = \delta_{j,k} t$.*

Proof: The forward implication is clear and it suffices to prove the reverse implication. So, suppose that each $M_j(\cdot)$ is a local $(\mathcal{F}_t)$ martingale and that $\langle M_j, M_k\rangle_t = \delta_{jk} t$. Let $\xi \in \mathbb{R}^d$ and define

$$W_t = \exp\left(i\sum_{l=1}^d \xi_j M_l(t) - \frac{1}{2}|\xi|^2 t\right).$$

We would like to apply Itô's formula to W. However, W is a complex valued function of $(M_1(t), \ldots, M_d(t))$ and as such is (slightly) beyond what the complex versions of Itô's formula stated in the previous chapter can handle. Instead, we write

$$\begin{aligned}
W_t &= \exp\left(-\frac{1}{2}|\xi|^2 t\right) \cos\left(\sum_{l=1}^d \xi_l M_l(t)\right) + i \exp\left(-\frac{1}{2}|\xi|^2 t\right) \sin\left(\sum_{l=1}^d \xi_l M_l(t)\right) \\
&= V_t \cos(U_t) + iV_t \sin(U_t)
\end{aligned}$$

(where $U_t = \sum_{l=1}^d \xi_l M_l(t) - \frac{1}{2}|\xi|^2 t$ and $V_t = e^{-\frac{1}{2}|\xi|^2 t}$) and apply the real Itô formula componentwise to the real and imaginary parts. Note that this is an entirely rigorous approach; the Itô integral extends to higher dimensions by componentwise operations and $\mathbb{C} \cong \mathbb{R}^2$. The result is

$$\begin{aligned} W_t &= \sum_{j=1}^d \int_0^t -\xi_j V_s \sin(U_s) dM_j(s) + \frac{1}{2} \sum_{j,k=1}^d \int_0^t -\xi_j^2 V_s \cos(U_s) d\langle M_j, M_k \rangle_s + \int_0^t \frac{1}{2}|\xi|^2 V_s \cos(U_s) ds \\ &\quad + i \left\{ \sum_{j=1}^d \int_0^t \xi_j V_s \cos(U_s) dM_j(s) + \frac{1}{2} \sum_{j,k=1}^d \int_0^t -\xi_j^2 V_s \sin(U_s) d\langle M_j, M_k \rangle_s + \int_0^t \frac{1}{2}|\xi|^2 V_s \sin(U_s) ds \right\} \\ &= \sum_{j=1}^d \int_0^t i\xi_j V_s e^{iU_s} dM_j(s) \end{aligned}$$

(note that the two rightmost terms cancel in both of the above lines). It follows immediately that W_t is a complex local martingale. Since $|W_t| \leqslant 1$ it follows immediately by Lemma 1.4.2 that W_t is in fact a martingale. Therefore, by the optional stopping theorem we have $\mathbb{E}[W_t - W_s \mid \mathcal{F}_s] = 0$ for $s \leqslant t$, which is precisely (2.7). It follows immediately by Lemma 2.3.1 that M_t is a Brownian motion. ■

Corollary 2.3.4 *Let Z be a $\mathbb{C}$ valued process and suppose Z is adapted to the filtration $(\mathcal{F}_t)$. Then Z is a Brownian motion if and only if all the following conditions hold:*

1. *Z almost surely has continuous paths.*
2. *Z is a conformal local martingale*
3. *$\langle Z, \bar{Z} \rangle_t = 2t$ for all t.*

Proof: The forwards implication follows from using Lemma 2.2.8 and noting that $\langle Z, \bar{Z} \rangle_t = \langle X \rangle_t + \langle Y \rangle_t - 2i \langle X, Y \rangle_t = 2t$. For the reverse implication,

$$\begin{aligned} 0 &= \langle X \rangle_t - \langle Y \rangle_t + 2i \langle X, Y \rangle_t \\ 2t &= \langle X \rangle_t + \langle Y \rangle_t - 2i \langle X, Y \rangle_t. \end{aligned}$$

Solving the above gives $\langle X \rangle_t = \langle Y \rangle_t = t$ and $\langle X, Y \rangle = 0$. The result then follows from Theorem 2.3.3. ■

It is natural to believe that complex Brownian motion is preserved (with no need for time change) under rotations. Thanks to the Lévy characterisation this is very easy to prove.

Lemma 2.3.5 *Let Z be a complex Brownian motion with $Z_0 = 0$ and let $f(z) = ze^{i\theta}$ where $\theta \in [0, 2\pi)$. Then $t \mapsto f(Z_t)$ is a complex Brownian motion.*

Proof: Since Z and $f(z) = e^{i\theta} z$ are continuous, $e^{i\theta} Z$ is continuous. Since f is deterministic, $e^{i\theta} Z$ is adapted to the same filtration as Z and since f is linear $e^{i\theta} Z$ is a martingale. The bilinearity of the bracket process gives

$$\begin{aligned} \langle e^{i\theta} Z, e^{i\theta} Z \rangle_t &= e^{2i\theta} \langle Z, Z \rangle_t = 0 \\ \langle e^{i\theta} Z, \overline{e^{i\theta} Z} \rangle_t &= e^{i(\theta - \theta)} \langle Z, \bar{Z} \rangle_t = 2t \end{aligned}$$

and it follows from Corollary 2.3.4 that $e^{i\theta}Z$ is a complex Brownian motion. ■

The remainder of this section is concerned with time changes. In words, a time change means that we reparameterize the time coordinate, in much the same way as is standard for ODEs. In the random setting there is the extra freedom that we may change time differently along each path of the random process.

Definition 2.3.6 *Let $(\mathcal{F}_t)$ be a filtration and let M be an $(\mathcal{F}_t)$ adapted process. An $(\mathcal{F}_t)$* ***time change*** *is a collection $(\tau_t)_{t=0}^{\infty}$ of finite stopping times such that*

- *for each s, τ_s is an $(\mathcal{F}_t)$ stopping time and*
- *for all $s < t$, $\tau_s \leqslant \tau_t$.*

The time change $\tau = (\tau_t)$ is said to be strictly increasing if $\tau_s < \tau_s$ whenever $s < t$.

We use τ_t and $\tau(t)$ interchangeably. The process M time changed by τ is $t \mapsto M_{\tau(t)}$ and is adapted to the filtration $(\mathcal{F}_{\tau(t)})$.

Lemma 2.3.7 *Let Z be a complex conformal $(\mathcal{F}_t)$ martingale with continuous paths. Suppose that $t \mapsto \langle Z, \bar{Z}\rangle_t$ is strictly increasing and that $\lim_{t\to\infty}\langle Z, \bar{Z}\rangle_t = \infty$. Define*

$$\tau_t = \inf\{s \geqslant 0\,;\, \langle Z, \bar{Z}\rangle_s > 2t\}.$$

Then $\tau = (\tau_t)$ is a strictly increasing time change and $t \mapsto Z_{\tau(t)}$ is a complex Brownian motion adapted to $(\mathcal{F}_{\tau(t)})$.

Proof: Note that $t \mapsto \langle Z, \bar{Z}\rangle_t$ is automatically non-negative and continuous (and also increasing, but our hypothesis is stronger). Since $t \mapsto \langle Z, \bar{Z}\rangle_t$ is strictly increasing and continuous, $t \mapsto \tau_t$ is also strictly increasing and continuous. Since $\langle Z, \bar{Z}\rangle_t \to \infty$ as $t \to \infty$, $\tau(t) < \infty$ for all t. Hence $t \mapsto Z_{\tau(t)}$ is continuous. Further, since $t \mapsto \langle Z, \bar{Z}\rangle_t$ is $(\mathcal{F}_t)$ adapted, for each s τ_s is an $(\mathcal{F}_t)$ stopping time.

Define

$$\sigma(n) = \inf\{t \geqslant 0\,;\, |Z_{\tau(t)}| \geqslant n\}, \quad \phi(n) = \frac{1}{2}\langle Z, \bar{Z}, \rangle_{\sigma(n)}.$$

Note that

$$\begin{aligned}\tau(\phi(n)) &= \inf\{s > 0\,;\, \langle Z, \bar{Z}\rangle_s > 2\phi(n)\}\\ &= \inf\{s > 0\,;\, \langle Z, \bar{Z}\rangle_s > \langle Z, \bar{Z}\rangle_{\sigma(n)}\}\\ &= \sigma(n)\end{aligned}$$

where the last line follows by continuity of $\langle Z, \bar{Z}\rangle_\cdot$. Since τ is strictly increasing we have

$$\{t \leqslant \phi(n)\} = \{\tau(t) \leqslant \sigma(n)\} \tag{2.8}$$

and hence

$$\tau(t \wedge \phi(n)) = \sigma(n) \wedge \tau(t). \tag{2.9}$$

By (2.8), $\phi(n)$ is an $(\mathcal{F}_{\tau(t)})$ stopping time and by definition of $\sigma(n)$ the process $t \mapsto Z_{t\wedge\sigma(n)}$ is a bounded martingale. Applying the optional stopping theorem and (2.9) we have

$$\mathbb{E}\left[Z_{\tau(t\wedge\phi(n))}\,|\,\mathcal{F}_{\tau(s)}\right] = \mathbb{E}\left[Z_{\sigma(n)\wedge\tau(t)}\,|\,\mathcal{F}_{\tau(s)}\right] = Z_{\sigma(n)\wedge\tau(s)} = Z_{s\wedge\phi(n)} \tag{2.10}$$

for all $s \leqslant t$. Since $|Z|$ is continuous we have $\sigma(n) \uparrow \infty$ as $n \to \infty$, hence $Z_{\tau(t)}$ is a local $(\mathcal{F}_{\tau(t)})$ martingale. Hence $\bar{Z}_{\tau(t)}$ is also an $(\mathcal{F}_{\tau(t)})$ martingale. Hence $Z_{\tau(t)}\bar{Z}_{\tau(t)} - \langle Z_{\tau(\cdot)}, \bar{Z}_{\tau(\cdot)}\rangle_t$ is an $(\mathcal{F}_{\tau(t)})$ local martingale and in particular the bracket $\langle Z_{\tau(\cdot)}, \bar{Z}_{\tau(\cdot)}\rangle$ is well defined. Hence

$$\langle Z_{\tau(\cdot)}, \bar{Z}_{\tau(\cdot)}\rangle_t = \langle Z, \bar{Z}\rangle_{\tau(t)} = 2t$$

by definition of τ and continuity of the bracket. By Lemma 2.2.3 we have $\langle Z_{\tau(\cdot)}, \bar{Z}_{\tau(\cdot)}\rangle_t = 2t$. Further, since Z is conformal we have $\langle Z, Z\rangle = 0$ and hence also $\langle Z_{\tau(\cdot)}, Z_{\tau(\cdot)}\rangle = 0$, so $Z_{\tau(\cdot)}$ is conformal.

We thus have all the conditions with which to apply Corollary 2.3.4 to Z, from which it follows that Z is a complex Brownian motion. ■

Remark 2.3.8 *An alternative proof of Lemma 2.3.5 is to use Lemma 2.3.7 and show that in this case $\tau_t = 2t$.*

Corollary 2.3.9 (Dubins-Schwarz) *Let M be a continuous real valued local martingale and suppose that $\langle M\rangle_t \to \infty$ as $t \to \infty$. Define*

$$\tau_t = \inf\{s \geqslant 0\,;\, \langle M\rangle_s > t\}.$$

Then $t \mapsto M_{\tau(t)}$ is a Brownian motion adapted to $(\mathcal{F}_{\tau(t)})$.

Sketch of Proof: Corollary 2.3.9 can be proved directly in similar style to our proof of Theorem 2.3.7, using Theorem 2.3.3 in place of Corollary 2.3.4. We omit the details, save for the comment that in the real case the bracket need not be strictly increasing. ■

2.4 Recurrence

In this section we prove that complex Brownian motion is recurrent, in as strong a sense as one could reasonably expect of a random process in a continuum.

Lemma 2.4.1 *Let Z be a complex Brownian motion with $Z_0 = 0$ and let $R > 0$. Then*

$$\mathbb{P}\left[\exists t, |Z_t| \geqslant R\right] = 1.$$

Proof: Let $Z = X + iY$. Then (X, Y) is a Brownian motion in $\mathbb{R}^2$ and in particular at time t has distribution $\mathcal{N}(0, t)$. Hence,

$$\mathbb{P}\left[|Z_t| \leqslant R\right] = \int_0^R \frac{1}{2\pi t} e^{-r^2/2t} 2\pi r\, dr = \left[-e^{-r^2/2t}\right]_0^R = 1 - e^{-r^2/2t} \leqslant \frac{R^2}{2t}.$$

Therefore,

$$\sum_{n=1}^{\infty} \mathbb{P}\left[|Z_{n^2}| \leqslant R\right] \leqslant \sum_{n=1}^{\infty} \frac{R^2}{2n^2} < \infty$$

and by Borel-Cantelli, $\mathbb{P}\left[|Z_{n^2}| \leqslant R \text{ i.o.}\right] = 0$, which implies the stated result. ■

Let A be an open annulus containing 0 and such that

$$A = \{z \in \mathbb{C}\,|\,, r < |z - w| < R\} \tag{2.11}$$

for some $w \in \mathbb{C}$ (w is the center of the annulus). Let $T_A = \inf\{t \geqslant 0\,;\, Z_t \notin A\}$ and note that Lemma 2.4.1 implies that $T_A < \infty$ almost surely.

Lemma 2.4.2 *Let Z be a complex Brownian motion with $Z_0 = 0$ and let T_A be the first exit time of Z from A. Then $T_A < \infty$ almost surely and*

$$\mathbb{P}\left[|Z_{T_A} - w| = r\right] = \frac{\log R - \log|w|}{\log R - \log r}$$
$$\mathbb{P}\left[|Z_{T_A} - w| = R\right] = \frac{\log|w| - \log r}{\log R - \log r}$$

Proof: Let $w_0 = u_0 + iv_0$, $Z_t = X_t + iY_t$ and set $R_t^2 = |Z_t - w|^2 = (X_t - u_0)^2 + (Y_t - v_0)^2$. Then Itô's formula shows that

$$\begin{aligned} dR_t^2 &= 2(X_t - u_0)dX_t + 2(Y_t - v_0)dY_t + 2dt \\ &= 2R_t dBt + 2dt \end{aligned}$$

where

$$dB_t = \frac{1}{R_t}\left((X_t - u_0)dX_t + (Y_t - y_0)dY_t\right).$$

Hence B_t is a local martingale and

$$\langle B, B\rangle_t = \int_0^t \frac{1}{R_s^2}\left((X_t - u_0)^2 + (Y_t - v_0)^2\right) ds = t$$

so by Theorem 2.3.3 B is a (real) Brownian motion. A further application of Itô's formula gives

$$d\log R_t^2 = \frac{2}{R_t} dB_t$$

and hence

$$\log|Z_t - w| = \log|w| + \int_0^t \frac{dB_t}{|Z_s - Z_0|}$$

is a local martingale. Further, by definition of T_A, $\log|Z_{t\wedge T_A} - w|$ is a bounded local martingale and thus a martingale by Lemma 1.4.2. By the Optional Stopping Theorem,

$$\begin{aligned} \log|w| &= \mathbb{E}\left[\log|Z_{T_A} - w|\right] \\ &= \mathbb{P}\left[|Z_{T_A} - Z_0| = r\right]\log r + \mathbb{P}\left[|Z_{T_A} - w| = R\right]\log R. \end{aligned}$$

Further, since $T_A < \infty$ we have

$$\mathbb{P}\left[|Z_{T_A} - w| = r\right] + \mathbb{P}\left[|Z_{T_A} - w| = R\right] = 1.$$

We thus have a pair of linear equations which, when solved, complete the proof. ■

Lemma 2.4.3 *Let Z be a complex Brownian motion with $Z_0 = 0 \in \mathbb{C}$. Then for every $w \in \mathbb{C}$ and $\epsilon > 0$,*

$$\mathbb{P}\left[\exists t > 0, Z_t \in \mathcal{B}(w, \epsilon)\right] = 1.$$

Proof: It $w = z_0$ then we are done. If not, by Lemma 2.4.2 we have

$$\mathbb{P}\left[\exists t, Z_t \in \mathcal{B}(w, \epsilon)\right] \geqslant \frac{\log R - \log|w|}{\log R - \log r}.$$

Letting $R \to \infty$, we have the stated result. ■

Theorem 2.4.4 *Let Z be a complex Brownian motion with $Z_0 = z_0 \in \mathbb{C}$. Then the closure of the range of Z is almost surely equal to $\mathbb{C}$.*

PROOF: Let (d_n) be a countable dense sequence in $\mathbb{C}$. Then by Lemma 2.4.3,

$$\mathbb{P}\left[\forall n, m \in \mathbb{N}, \exists t, Z_t \in \mathcal{B}(d_n, 1/m)\right] = 1.$$

For any $z \in \mathbb{C}$ there exists a subsequence (d_{r_n}) of (d_n) such that $d_{r_n} \to z$, and since for all n there exists t_n such that $Z_{t_n} \in \mathcal{B}(d_{r_n}, 1/n)$ we have $Z_{t_n} \to z$. ■

2.5 Conformal Invariance

In view of Lemma 2.3.5 it is natural to ask whether $f(Z_t)$ is a Brownian motion for a wider class of functions f than just rotations. In fact, seeing Lemmas 2.3.4 and 2.3.7 together with Itô's formula should suggest that the natural situation in which to ask this question is when we further permit the process $f(Z_t)$ to be time changed. We thus arrive at the *conformal invariance of complex Brownian motion.*

Theorem 2.5.1 (Lévy) *Let f be a non-constant entire function and let Z be a conformal local martingale. Suppose that $\langle Z, \bar{Z}\rangle_\cdot$ is strictly increasing. Then there exists a strictly increasing time change τ such that $f(Z_{\tau(\cdot)})$ is a complex Brownian motion.*

Remark 2.5.2 *In particular, entire functions preserve precisely the irregular nature in which Brownian paths oscillate.*

PROOF: By Itô's formula (in particular, Corollary 2.2.10) $f(Z_t)$ is a continuous conformal local martingale and

$$\langle f(Z_\cdot), \overline{f(Z_\cdot)}\rangle_t = \int_0^t |f'(Z_s)|^2 ds. \tag{2.12}$$

The stated result then follows from Lemma 2.3.7 providing we can show that (2.12) is strictly increasing and tends to infinity as $t \to \infty$.

To see that (2.12) is strictly increasing, note f' is an entire function that is not identically zero and hence (by the identity theorem) the set of zeros of f' cannot have a limit point. Since any uncountable subset of $\mathbb{C}$ has a limit point, f' can have at most countably many zeros. For any $z_0 \in \mathbb{C}$ such that $f'(z_0) = 0$ and any $s < t$,

$$\mathbb{E}\left[\int_s^t \mathbb{1}\{Z_u = z_0\} du\right] = \int_s^t \mathbb{P}\left[Z_u = z_0\right] du = 0.$$

Hence, almost surely, for almost all $u \in (s, t)$ we have $f'(Z_u) \neq 0$ and hence $\int_s^t |f'(Z_u)|^2 du > 0$. It follows that (2.12) is strictly increasing in t.

It remains to show convergence to infinity. Since f is non-constant, there exists an open ball $\mathcal{B}(w, \delta)$ and some $\epsilon > 0$ such that $|f'(z)| \geqslant \epsilon$ for all $\mathcal{B} \in B(w, \delta)$. Let B be a complex Brownian motion. By continuity of B there exists $\kappa > 0$ such that

$$\mathbb{P}_w\left[\sup_{s \in [0,\kappa]} |B_s - w| < \delta/3\right] > 0. \tag{2.13}$$

We construct a sequence (T_n) of stopping times according to the following procedure: Time T_1 occurs when Z first enters $\mathcal{B}(w,\delta/3)$. We then wait until time $T_1+\kappa$. Then, T_2 occurs at the next time at which Z enters $\mathcal{B}(w,\delta/3)$. We repeat this procedure ad infinitum to define T_n for all n. Note that Theorem 2.4.4 and the strong Markov property imply that $T_n<\infty$ for all n. Further, by the strong Markov property the events

$$A_n=\{Z_s\in\mathcal{B}(Z_{T_n},\delta/3)\text{ for all }s\in[T_n,T_n+\kappa)\}$$

are mutually independent. By (2.13) the probability of such events occur is bounded away from zero, so (by Borel-Cantelli) almost surely infinitely many A_n occur. Since, on each such n, $\mathcal{B}(Z_{T_n},\delta/3)\subseteq\mathcal{B}(w,\delta)$, we have

$$\int_{T_n}^{T_n+\kappa}|f'(Z_s)|^2ds\geqslant\kappa\epsilon^2$$

for infinitely many n. It follows that (2.12) tends to ∞ as $t\to\infty$. ■

In Theorem 2.4.4 we showed that the path of complex Brownian motion was almost surely dense in the complex plane. In contrast, with Theorem 2.5.1 we can show that the region of the complex plane that the path actually covers is rather small.

Lemma 2.5.3 *Let Z_t be a complex Brownian motion with $Z_0=z_0\in\mathbb{C}$ and let $w\neq z_0$. Then*

$$\mathbb{P}[\exists t,Z_t=w]=0.$$

PROOF: Suppose first that $Z_0=0$. Since $z\mapsto e^z$ is analytic, by Theorem 2.5.1 there is a time change τ such that $W_t=\exp(f(Z_{\tau(t)}))$ is a complex Brownian motion. However, $z\mapsto e^z$ omits zero (we proved this in Lemma 3.1.3) so $\mathbb{P}[\exists t,W_t=0]=0$. Since $W_0=1$ we have proved the result for the case $z_0=1$ and $w=0$.

For arbitrary $z_0\neq w$, the full result can be deduced by applying a Moebius transformation f to W, such that $f(1)=z_0$, $f(0)=w$ and $f(\infty)=\infty$. Then f is analytic and there is a time change τ' such that $W'_t=f(W_{\tau'(t)})$ is a Brownian motion, but of course W' never hits w. ■

Lemma 2.5.4 *Let Z be a complex Brownian motion with $Z_0=0$. Then $\{z\,;\,\exists t,Z_t=z\}$ is almost surely a Lebesgue-null subset of $\mathbb{C}$.*

PROOF: Let $A=\{z\,;\,\exists t,Z_t=z\}$. Then, using Fubini's Theorem,

$$\mathbb{E}\left[\int_{\mathbb{C}}\mathbb{1}\{z\in A\}dz\right]=\int_{\mathbb{C}}\mathbb{P}[\exists t,Z_t=z\in]\,dz=0$$

by Lemma 2.5.3, and since $\mathbb{1}\{z\in A\}\geqslant0$ we conclude that $\mathbb{P}\left[\int_{\mathbb{C}}\mathbb{1}\{z\in A\}dz=0\right]=1$. ■

Null subsets are rarely boring, especially when they occur naturally and this case is no exception. It is beyond the scope of this course (and requires different tools) but in fact the range of planar Brownian motion is a fractal with Hausdorff dimension 2. Therefore, Lemma 2.5.4 shows that even once we have the right dimension the corresponding Hausdorff measure is zero.

An even more sensitive tool than the usual Hausdorff measure is required to properly identify the fractal nature of complex Brownian paths. To be precise, for planar Brownian motion we must use generalized Hausdorff measure with gauge function

$$\phi(t)=t^2\log\log t$$

and this defines a measure under which finite time intervals of Brownian paths have finite non-zero area[1]

[1]If this means nothing to you, using $\phi(t) = t^2$ as the gauge function gives 2 dimensional Hausdorff measure; the exponent corresponds to change in measure when length is scaled.

Chapter 3

Winding and Tangling

Recall that the function f is said to omit the point z if z is not an element of the range of f. The final result of this chapter (and of this course) will be Picard's (Little) Theorem; a non-constant entire function may omit at most a single point from its range.

This is a course about probability and we will give a probabilistic proof Picard's Theorem. In particular, we will prove Picard's Theorem using the winding and tangling of Brownian paths. That said, the statement of Picard's Theorem belongs firmly in the realm of complex analysis; before we embark on the road towards its proof let us view the statement of Picard's Theorem in its proper light.

3.1 Picard's Theorem in Complex Analysis

The branch of complex analysis that is concerned with studying the range of entire functions is known as Nevanlinna theory. The 'first theorem' in Nevanlinna theory is the following well known result.

Theorem 3.1.1 (Louville) *Let f be an entire function and suppose f is bounded. Then f is constant.*

Louville's Theorem, which is usually proved using Taylor's Theorem, is in fact a much weakened version of Picard's Theorem; it says that if an entire function omits $\{z \in \mathbb{C}\,;\,|z| \geqslant M\}$ from its range then f must be constant.

A result which will not be of direct use to us but which is of interest to us, is the complex version of the fundamental theorem of algebra:

Theorem 3.1.2 *Let $P(z) = \sum_{i=0}^{n} z^i a_i$ be a complex polynomial. Then, for each $w \in C$ the equation $P(z) = w$ has precisely n solutions in $\mathbb{C}$ (counted by multiplicity).*

The connection to Picard's Theorem should be clear: polynomials are perhaps the most prominent examples of entire functions and an immediate corrolary of the above result is that polynomials do not omit *any* values.

In fact, more is true. A deeper result, which generalizes both Theorem 3.1.2 and Picard's theorem (and is not part of this course!), shows that an entire function might omit a single value but, amongst the values that it does take, each value is taken essentially the same number of times.

In view of Theorem 3.1.2, one might wonder how easy it is to give an example of an entire function that really does omit some point.

Lemma 3.1.3 *The function $f(z) = e^z$ is entire and omits* 0.

PROOF: Recall that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. It is a theorem that a complex power series defines an analytic function within its radius of convergence, so f is entire. From the power series it is trivial to see that $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$. To finish, if $w \in \mathbb{C}$ and $e^w = 0$ then we have $1 = e^0 = e^{-w}e^w = 0$, which is a contradiction. ■

3.2 Winding

In this section we will collect together some definitions and theorems from complex analysis that make precise the concept of a curve winding around a point.

Definition 3.2.1 *A connected open subset of $\mathbb{C}$ is called a **domain**.*

Definition 3.2.2 *Let D be a domain. For $a, b \in \mathbb{R}$ with $a < b$, a continuous function $\gamma : [a, b] \to D$ is known as a **path** in D. The path γ is said to be **closed** if $\gamma(a) = \gamma(b)$ and **simple** if $\gamma(s) \neq \gamma(t)$ for all $a < s < t < b$.*

*A continuously differentiable path is known as a **curve**. A closed curve γ is said to be **contractible** in a domain D if γ can be continuously deformed within D to a constant path.*

We use the notation $\gamma(t)$ and γ_t interchangeably.

Example 3.2.3 *Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then γ is a simple closed curve that is contractible in $\mathbb{C}$, but not contractible in $\mathbb{C}\backslash\{0\}$. Further, $\gamma_1 = \gamma|_{[0,\pi]}$ is a simple curve that is contractible in $\mathbb{C}$ and in $\mathbb{C}\backslash\{0\}$, but is not closed.*

If a path $\gamma : [a, b] \to \mathbb{C}$ does not pass through $z_0 \in \mathbb{C}$ then we can define a function $\theta : [a, b] \to \mathbb{R}$ by

$$\gamma(t) = z_0 + |\gamma(t) - z_0| e^{i\theta(t)},$$

with the additional requirement that θ is continuous. The function θ is said to be a continuous choice of the argument of γ about z_0. Note that many such choices of θ are possible; just add integer multiples of 2π.

For $a \leqslant s < t \leqslant b$, the quantity $\theta(t) - \theta(s)$ measure the angle (relative to z_0, cumulatively and in the anticlockwise direction) through which γ turns during $[s, t]$.

Definition 3.2.4 *If $\gamma : [a, b] \to \mathbb{C}$ is a path and θ is a continuous choice of its argument about z_0 then we say*

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

*is the **winding number** of γ about z_0.*

Remark 3.2.5 *Definition 3.2.4 did not require that γ to be a curve (i.e. differentiable). It is straightforward to show that the winding number does not depend on the particular continuous choice of argument used.*

We will write the complex path integral of the function $f : D \to \mathbb{C}$ along the curve $\gamma : [a, b] \to D$ as

$$\int_\gamma f(\gamma)d\gamma = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \Re\big(f(\gamma(t))\big)\gamma'(t)dt + i\int_a^b \Im\big(f(\gamma(t))\big)\gamma'(t)dt,$$

where the real numbers $\Re(w)$ and $\Im(w)$ respectively denote the real and imaginary part of $w \in \mathbb{C}$. We are now in a position to recall what is probably the most famous result in complex analysis.

Theorem 3.2.6 (Cauchy) *Let D be a domain and let γ be a simple closed curve in D that is contractible in D. Let f be analytic in D. Then*

$$\int_\gamma f(\gamma)d\gamma = 0.$$

Let $\gamma : [0, t] \to \mathbb{C}$ be a curve such that γ does not pass through $w \in \mathbb{C}$. For $0 \leqslant s \leqslant t$ we define

$$\theta(s) = \int_0^s \frac{1}{\gamma - w} d\gamma. \tag{3.1}$$

Lemma 3.2.7 *For all s, $\Im\theta$ is a continuous choice of the argument of γ about w.*

Proof: We write $\gamma(s) = w + r(s)e^{i\phi(s)}$, for real functions r and ϕ. Note that continuity of γ implies continuity of both r and ϕ. Then

$$d\gamma(s) = e^{i\phi(s)}dr(s) + ir(s)e^{i\phi(s)}d\phi(s)$$

by the product rule. Hence,

$$\frac{1}{\gamma(s) - w}d\gamma(s) = \frac{1}{r(s)}dr(s) + i\,d\phi(s),$$

which implies that

$$\int_0^s \frac{1}{\gamma(u) - w}d\gamma(u) = \int_0^s \frac{1}{r(u)}dr(u) + i(\phi(s) - \phi(0)). \tag{3.2}$$

Hence $\Im\theta(s) = \phi(s) - \phi(0)$, which completes the proof. ■

Lemma 3.2.8 *A closed path γ in $\mathbb{C}\backslash\{0\}$ is contractible in $\mathbb{C}\backslash\{0\}$ if and only if the winding number of γ about 0 is 0.*

Proof:

■

3.3 Winding of Brownian Paths

We have already seen that windings numbers of curves can be expressed as complex path integrals, but the expression (3.1) does not make sense if γ is not a curve. We are already concious that Brownian paths are far from being curves; they fail to be differentiable and, worse still, do not have finite variation. Consequently, we must seek an alternative for (3.1) through the Itô integral.

Lemma 3.3.1 *Suppose that Z_t is a complex Brownian motion with $Z_0 = z_0 \neq 0$ and define*

$$\Theta_t = \int_0^t \frac{1}{Z_s} dZ_s.$$

Then $\Im\Theta_t$ is a continuous choice of the argument about 0 of the path of Z over $[0, t]$.

PROOF: By Itô's formula (in particular, Corollaries 2.2.10 and 2.2.11),

$$\begin{aligned} d(Z_t \exp(-\Theta_t)) &= \exp(-\Theta_t) dZ_t + Z_t d(\exp(-\Theta_t)) \\ &= \exp(-\Theta_t) dZ_t - Z_t \exp(-\Theta_t) d\Theta_t \\ &= \exp(-\Theta_t) dZ_t - Z_t \exp(-\Theta_t) \frac{1}{Z_t} dZ_t \\ &= 0. \end{aligned}$$

To be precise, in the above calculation we use that $\mathbb{P}[\exists t, Z_t = 0] = 0$, which follows from Lemma 2.5.3, and take the function $z \mapsto \frac{1}{z}$ to have domain $D = \mathbb{C}\backslash\{0\}$. Hence $Z_t = Z_0 \exp(\Theta_t) = Z_0 \exp(\Re\Theta_t) \exp(\Im\Theta_t)$ and the result follows. ■

The proof of the above lemma reveals an important technique that has been waiting for us since Corollary 2.2.10. The Itô calculus of conformal local martingales and analytic functions follows essentially the same rules as the deterministic calculus of real valued smooth functions. Consequently, the paths of conformal local martingales, despite being decidedly rough (see Remark 2.2.4), sometimes behave in much the same way as their differentiable relatives.

Theorem 3.3.2 *Let Z_t be a complex Brownian motion with $Z_0 = z_0 \neq 0$ and let M_t be the winding number of the path of Z during $[0, t]$ about 0. Then there exists an increasing sequence (T_n) of times such that $M_{T_n} = 0$ and $T_n \to \infty$ almost surely as $N \to \infty$.*

PROOF: By Lemma 3.3.1 the process

$$M_t = \Im\left(\int_0^t \frac{1}{Z_s} dZ_s\right)$$

is both a continuous choice of the argument about 0 of Z and a (real) continuous local martingale. Further, without loss of generality (or consider $e^{-\Im Z_0} Z$ and apply Lemma 2.3.5) we can assume $z_0 \in \mathbb{R}$ so as $\frac{1}{2\pi} M_t$ is the winding number about 0 of the path Z during $[0, t]$. Further,

$$\langle M \rangle_t = \int_0^t \frac{1}{|Z_s|^2} ds$$

and (a one dimensional version of) the argument used in the proof of Theorem 2.5.1 shows that $\langle M \rangle_t \to \infty$ as $t \to \infty$. Hence, by Corollary 2.3.9, there is a time change τ such that $W_t = M_{\tau(t}$ is a real Brownian motion. Further, since $\frac{1}{|Z_s|^2} > 0$, $\langle M \rangle$ is strictly increasing and hence τ is strictly increasing. Of course, there is an increasing sequence of times (S_n) such that $S_n \to \infty$ such that $W_{S_n} = 0$. Setting $T_n = \tau^{-1}(S_n)$ completes the proof. ■

3.4 Tangling of Brownian Paths

In fact, in order to prove Picard's Theorem it is not winding that is required, but tangling. Tangling is much the same as winding except that more points are involved.

Definition 3.4.1 *Let $D \subseteq \mathbb{C}$. A closed path $\gamma \subseteq D$ is said to be **tangled** in D if γ is not contractible in D.*

From hereon, if Z is a complex Brownian motion, let Z_t° denote the path of Z during time $[0, t]$, concatenated with a single line segment joining Z_t to Z_0, thus making Z_t° a closed path in $\mathbb{C}$.

The result we need, in contrast to Theorem 3.3.2, is the following. The proof constitutes the remainder of this section.

Theorem 3.4.2 (Itô, McKean) *Let Z be a complex Brownian motion and suppose $Z_0 = 0$. Then there exists a random time $\eta < \infty$ such that for all $s \geqslant \eta$, if $Z_s \in (-1, 1)$ then Z_s° is tangled in $\mathbb{C}\backslash\{-1, 1\}$.*

In words, complex Brownian motion eventually becomes tangled in $\mathbb{C}\backslash\{-1, 1\}$. Because of Theorem 3.3.2, we know that one which of 0 and 1 it winds around must change infinitely many times as $t \to \infty$. Consequently, Theorem 3.4.2 is a sensitive result and its proof, which constitutes the rest of this section, requires a significant amount of care.

Remark 3.4.3 *The original proof of Theorem 3.4.2 was given by Itô and McKean. Their argument, which uses Modular functions, appears in McKean (1969). The proof gives below is adapted from an argument due to Doob that appeared in Davis (1979).*

We will need to introduce a significant amount of notation in the following proof, since it involves a somewhat complicated encoding of planar paths. Such notation will only be used within this section.

Let G be the free group with two generators a and b. We denote the inverse of $c \in G$ by c^{-1}. Let $A = \{a, b, a^{-1}, b^{-1}\}$ and let W be set of words (i.e. finite ordered sets) with letters in A, including the empty word. We write $\xi\zeta$ to denote the concatenation of the words ξ and ζ. Then length of the word w, denoted by $|w|$, is the number of letters (including repeats) that make up w.

Definition 3.4.4 *For $\xi, \zeta \in W$ and $c \in A$, a word which has the form $\xi\zeta$ is said to be a **cancellation** of $\xi cc^{-1}\zeta$. We say the word w is a **simplification** of w' is there is a sequence of successive cancellations of w' that results in w.*

*For two words w_1 and w_2, we write $w_1 \sim w_2$ to mean that w_1 and w_2 have a common simplification. A word that has no simplifications is called a **reduced word**.*

Lemma 3.4.5 *The relation $\sim$ is an equivalence relation on W. For each equivalence class C there is a unique $c \in C$ such that $|c| \leqslant |d|$ for all $d \in C$. Each such c is a reduced word and as such is an element of G.*

PROOF: We omit the proof and leave it as an exercise. ■

We now set up a correspondence between paths in $\mathbb{C}\backslash\{-1, 1\}$ and words in W. Consequently, we achieve a map between paths in $\mathbb{C}\backslash\{-1, 1\}$ and reduced words. To this end, let $K = \mathbb{C}\backslash\{-1, 1\}$ and write

$$J_0 = (-\infty, -1)$$
$$J_1 = (-1, 1)$$
$$J_2 = (1, \infty)$$

and note that $K = J_0 \cup J_1 \cup J_2$. Let $\mathbb{H}^+ = \{z \in \mathbb{C}\,;\, \Im z > 0\}$ denote the open upper half plane and let $\mathbb{H}^- = \{z \in \mathbb{C}\,;\, \Im z < -\}$ denote the open lower half plane.

Let γ be a complex path in $\mathbb{C}\backslash\{-1,1\}$ started at some point $z \notin \mathbb{R}$. Let $T_0 = \inf\{t > 0\,;\, Z_t \in \mathbb{R}\}$ be the first time at which γ enters K. Then, by continuity of γ we have $\gamma(T_0) \in K$, so as $\gamma(T_0)$ is an element of precisely one of J_0, J_1 and J_2. We then define $(T_k)_{k\in\mathbb{N}}$ recursively. If $\gamma(T_k)$ is in section j at T_k then T_{k+1} is the next time at which $\gamma(t)$ hits $K\backslash J_j$.

To each passage during time $[T_{k-1}, T_k]$, we associate a triplet of information. The j such that $\gamma(T_{k-1}) \in J_j$ is known as the **source** and the j' such that $\gamma(T_k) \in J_{j'}$ is knows as the **sink**. The triplet is $(j, j', \star)$ where $\star \in \{-,+\}$ is defined as follows.

If there is some $\epsilon > 0$ such that $\gamma(t) \in \mathbb{H}^+$ for all $t \in [T_{k-1} - \epsilon, T_k)$ then we say γ hits $J_{j'}$ **from above** at T_k. If no such ϵ exists then there is some $\epsilon > 0$ such that $\gamma(t) \in \mathbb{H}^-$ for all $t \in [T_k - \epsilon, T_k)$, in which case we say γ hits $J_{j'}$ **from below** at T_k. If the passage is from above then $\star$ is $+$ and if the passage is from below then $\star$ is $-$.

Therefore, each finite path γ in $\mathbb{C}\backslash\{0,1\}$ maps to a finite sequence of triplets of information. Let us write $(\mathcal{I}^\gamma_k, \mathcal{O}^\gamma_k, \star^\gamma_k)_{k=1}^K$ for this sequence. We map this sequence onto a second sequence of words, which we write as $(\mathcal{W}^\gamma_k)_1^K$ where $\mathcal{W}^\gamma_k$ is defined as follows:

$(\mathcal{I}^\gamma_k, \mathcal{O}^\gamma_k, \star^\gamma_k)$	$\mathcal{W}^\gamma_k$
$(0,1,+)$	a
$(0,1,-)$	a^{-1}
$(1,0,+)$	a^{-1}
$(1,0,-)$	a
$(1,2,+)$	b
$(1,2,-)$	b^{-1}
$(2,1,+)$	b^{-1}
$(2,1,-)$	b
$(0,2,+)$	ab
$(0,2,-)$	$b^{-1}a^{-1}$
$(2,0,+)$	$b^{-1}a^{-1}$
$(2,0,-)$	ab

We define $\mathscr{W}(\gamma) = \mathcal{W}^\gamma_K \mathcal{W}^\gamma_{K-1} \ldots \mathcal{W}^\gamma_1$ to be the concatenation *from the left* of the words in $(\mathcal{W}^\gamma_k)$. Using Lemma 3.4.5, we define $\mathscr{G}(\gamma)$ to be the unique reduced word in the same $\sim$ equivalent class as $\mathscr{W}(\gamma)$.

Remark 3.4.6 *The free group G is associative but not commutative.*

Despite the slew of terminology we are still well on the beaten track. Fix $t > 0$ and let $\mathscr{P}$ be the set of closed paths $\kappa : [0,t] \to \mathcal{C}\backslash\{0,1\}$ such that $\kappa(0) \in K$. The following theorem, which we will not prove, is a standard result from algebraic topology.

Theorem 3.4.7 *The map $\gamma \mapsto \mathscr{G}(\gamma)$ is a continuous function from $\mathscr{P}$ (equipped with the uniform topology) to the free group with generators a^2 and b^2.*

Further, paths $\gamma_1, \gamma_2 \in \mathscr{P}$ can be continuously deformed through $\mathbb{C}\backslash\{-1,1\}$ to each other if and only if $\mathscr{G}(\gamma_1) = \mathscr{G}(\gamma_2)$.

Exercise 3.4.8 *The reader should draw some tangled γ of their own and calculate $\mathscr{G}(\gamma)$.*

As a consequence, the above theorem says that continuous deformation of γ through $\mathbb{C}\backslash\{-1,1\}$ does not change $\mathscr{G}(\gamma)$.

Corollary 3.4.9 *A closed path $\gamma \in \mathscr{P}$ is not tangled in $\mathbb{C}\backslash\{0,1\}$ if and only if $\mathscr{G}(\gamma)$ is the empty word.*

PROOF: This follows immediately from Theorem 3.4.7 and Corollary 3.2.8. ∎

In view of Corollary 3.4.9, to prove Theorem 3.4.2 we must show that, with probability one, there is some time $\eta < \infty$ such that $\mathscr{G}(Z_s^\circ)$ is non-empty whenever $Z_s \in (-1,1)$.

Let $\epsilon \in (0,1)$. Without loss of generality (by Lemma 2.5.3) assume that Z_t never visits -1 or 1. Let (S_n) be the sequence of stopping times such that $S_0 = 0$ and

$$S_{n+1} = \inf\{t \geqslant S_n\,;\, \mathscr{G}(Z_t^\circ) \neq \mathscr{G}(Z_{S_n}^\circ) \text{ and } Z_t \in (-1,1)\}.$$

Remark 3.4.10 *Note that $\mathscr{G}(Z_t^\circ)$ is not defined if $Z_t \in (-\infty,1) \cup (1,\infty)$ because then $\mathscr{G}(Z_t^\circ)$ is not contained within $\mathbb{C}\backslash\{-1,1\}$.*

By Theorem 3.4.7, for each n we can write $\mathscr{G}(Z_{S_n}^\circ)$ uniquely as

$$\mathscr{G}(Z_{S_n}^\circ) = g_{l(n)}^{j_{l(n)}} \dots g_2^{j_2} g_1^{j_1} \tag{3.3}$$

where $g_i \in \{a^2, b^2\}$ are group elements, $j_i \in \{1,-1\}$ are groups actions (so as $g_i^{j_1} = (g_i)^{j_i}$) and $l(n) \in \mathbb{N} \cup \{0\}$ ($l(n) = 0$ corresponds to the empty word). By definition of our system of coding paths, we have that for all n,

$$|l(n) - l(n+1)| = 1. \tag{3.4}$$

In words, during time $[S_n, S_{n+1}]$, the path of Z makes precisely one more or one less turn around either -1 or 1. The crucial point for us is to show that, more often than not, this causes the path to become more tangled rather than less. One of precisely four things can happen during $[S_n, S_{n+1}]$:

Z winds clockwise about -1	add a^2 to $\mathscr{W}(Z^\circ)$.
Z winds anti-clockwise about -1	add a^{-2} to $\mathscr{W}(Z^\circ)$.
Z winds clockwise about 1	add b^2 to $\mathscr{W}(Z^\circ)$.
Z winds anti-clockwise about 1	add b^{-2} to $\mathscr{W}(Z^\circ)$.

Using the notation of (3.3) for $\mathscr{G}(Z_{S_n}^\circ)$, we would have $l(n+1) = l(n) - 1$ if and only if the turn that occurs during $[S_n, S_{n+1}]$ undoes the turn corresponding to $g_{l(n)}^{j_{l(n)}}$ (i.e. inverts the group element). Note, by the strong Markov property of Z, that $n \mapsto \mathscr{G}(Z_{S_n}^\circ)$ is Markov with respect to the filtration $\mathcal{F}_n = \sigma(Z_t\,;\, t \leqslant S_n)$, but the probabilities of the next transition depend on the position of Z_{S_n}.

Consider the transition $[S_n, S_{n+1}]$ and write $g_{l(n)}^{j_{l(n)}} = g$. By Problem 4 on Sheet 3 (i.e. symmetry) and the strong Markov property, the chance of the transition during $[S_n, S_{n+1}]$ corresponding to g^{-1} is the same as the probability that it corresponds to g. It follows that:

($\dagger_1$) On each transition $[S_n, S_{n+1}]$, the probability of $l(n+1) = l(n) - 1$ is at most 1/2.

The transition $[S_n, S_{n+1}]$ is said to be **special** if $Z_{S_n} \in (-\frac{1}{2}, \frac{1}{2})$. In this case, Question 4 on Problem Sheet 3, along with the strong Markov property, implies that each of the four possible transitions which could occur happens with probability bounded away from 0 (see picture!). This has two important consequences; firstly infinitely many transitions occur and secondly, since (by the above paragraph) these four transitions fall into two pairs and within each pair each transition occurs equally likely. Therefore, for some $\delta > 0$,

($\dagger_2$) On a special transition $[S_n, S_{n+1}]$, the probability that $l(n+1) = l(n) - 1$ is at most $1/2 - \delta$.

By Problem 5 on Sheet 5 and the strong Markov property, infinitely many special transitions occur. From this and ($\dagger_1$), ($\dagger_2$), it follows immediately (by standard results concerning integer valued random walks) that $l(n) \to \infty$ almost surely as $n \to \infty$. This says precisely that the length of $\mathscr{G}(Z^\circ_{S_n})$ tends to infinity almost surely. For any t such that $Z_t \in (-\epsilon, \epsilon)$, if $n(t) = \sup\{s \leqslant t\,;\, s = S_n\}$ then we have $\mathscr{G}(Z^\circ_{S_{n(t)}}) = \mathscr{G}(Z^\circ_{S_n})$. By Corollary 3.4.9, Z°_t is only untangled if $\mathscr{G}(Z^\circ_{S_{n(t)}})$ is the empty word, which proves Theorem 3.4.2.

3.5 Picard's Theorem

We are now ready for the final step of course.

Theorem 3.5.1 (Picard) *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$. Then the range of f cannot omit both -1 and 1.*

Proof: Without loss of generality we can and will assume that $f(0) = 0$ and f omits both -1 and 1. (Else, let g be the (unique) Moebius transformation such that $g(f(0)) = 0, g(a) = -1, g(b) = 1$ and consider $g \circ f$.)

For the last time, let Z be a complex Brownian motion with $Z_0 = 0$. By Theorem 2.5.1 there is a time change τ such that $W_t = f(Z_{\tau(t)})$ is a complex Brownian motion. Let W°_t denote the path of W over $[0, t]$, followed by the line segment $[W_t, 0]$ to make a closed path. By Theorem 3.4.2 there is some $\eta < \infty$ such that W°_t is tangled in $\mathbb{C}\backslash\{-1, 1\}$ for all $t \geqslant \eta$. By Problem 5 on Sheet 5 there is some random time T such that $\eta \leqslant T < \infty$ and $W_t \in (-1, 1)$.

Then path Z°_T is contractible in $\mathbb{C}$ (as indeed is any path) but its image W°_T under the continuous function f is not contractible in $f(\mathbb{C}) \subseteq \mathbb{C}\backslash\{-1, 1\}$. This is a contradiction. ∎

Bibliography

B. Davis. Brownian motions and analytic functions. *Ann. Probab.*, 7:913–932, 1979.

S. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley, 1986.

H. P. McKean. *Stochastic Integrals*. Academic Press, New York, 1969.

P. Mörters and Y. Peres. *Brownian Motion*. Cambridge University Press, 2010.

H. Priestley. *Complex Analysis*. Oxford University Press, 2003.

L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales*. Cambridge University Press, 2000.

W. Rudin. *Real and Complex Analysis*. Mc, 1987.